

Vafa-Witten Invariants for Projective Surfaces I: Stable Case

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ABSTRACT. On a polarised surface, solutions of the Vafa-Witten equations correspond to certain polystable Higgs pairs. When stability and semistability coincide, the moduli space admits a symmetric obstruction theory and a \mathbb{C}^* action with compact fixed locus. Applying *virtual localisation* we define invariants constant under deformations.

When the vanishing theorem of Vafa-Witten holds, the result is the (signed) Euler characteristic of the moduli space of instantons. In general there are other, *rational*, contributions. Calculations of these on surfaces with positive canonical bundle recover the first terms of modular forms predicted by Vafa and Witten.

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1. INTRODUCTION

Fix a Riemannian 4-manifold S ; from the next Section onwards it will be a complex projective surface (with a Kähler metric). Fix a compact Lie group G and principal G -bundle $P \rightarrow S$. We let \mathcal{A}_P denote the set of all G -connections on P , and $\Omega^i(\mathfrak{g}_P)$ the \mathbb{C}^∞ i -forms on S with values in the adjoint bundle of P . Finally $\Omega^+ \subset \Omega^2$ denotes the self-dual 2-forms with respect to the Riemannian metric.

1.1. The Vafa-Witten equations. The Vafa-Witten equations [VW] are

$$(1.1) \quad \begin{aligned} \mathcal{A}_P \times \Omega^+(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) &\longrightarrow \Omega^+(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \\ (d_A, B, \Gamma) &\longmapsto (F_A^+ + [B, B] + [B, \Gamma], d_A \Gamma + d_A^* B). \end{aligned}$$

Here $[B, B]$ is defined by Lie bracket on \mathfrak{g}_P and the contraction $(T^*)^{\otimes 2} \otimes (T^*)^{\otimes 2} \rightarrow (T^*)^{\otimes 2}$ given by the (inverse of the) metric on the second and third T^* factors, followed by antisymmetrisation.

1.2. The equations on a Kähler surface. When (S, ω) is a Kähler surface, we can rewrite B and Γ in terms of a \mathfrak{g}_P -valued $(2, 0)$ -form $\phi \in$

$\Omega^{2,0}(\mathfrak{g}_P \otimes \mathbb{C})$ and a \mathfrak{g}_P -valued multiple of ω , resulting in

$$(1.2) \quad \begin{aligned} F_A^{0,2} &= 0, \\ F_A^{1,1} \wedge \omega + [\phi, \bar{\phi}] &= c \cdot \text{id}_E \omega^2, \\ \bar{\partial}_A \phi &= 0. \end{aligned}$$

Here c is a topological constant. From now on we will restrict attention to the cases $G = U(r)$ or $G = SU(r)$, so that P is the frame bundle of a hermitian vector bundle E . Then $c = \pi i \int_S c_1(E) \wedge \omega / r \int_S \omega^2$.

Thus, by the first equation, $\bar{\partial}_A$ defines an integrable holomorphic structure on E in with respect to which ϕ is holomorphic. Since the second equation is a moment map for the gauge group action, one can expect an infinite dimensional Kempf-Ness theorem that solutions (modulo $U(r)$ or $SU(r)$ gauge transformations) are equivalent to *stable* holomorphic pairs (E, ϕ) (modulo $GL(r, \mathbb{C})$ or $SL(r, \mathbb{C})$ gauge transformations) for a suitable notion of stability.

1.3. Hitchin-Kobayashi correspondence. Such a result is indeed true when $(S, \mathcal{O}_S(1))$ is polarised [AG, Ta1] with integral Kähler form $h = c_1(\mathcal{O}_S(1))$. In the $U(r)$ case solutions are equivalent to *slope polystable Higgs pairs*

$$(1.3) \quad (E, \phi), \quad \phi \in \text{Hom}(E, E \otimes K_X);$$

for $SU(r)$ we fix $\det E = \mathcal{O}_S$ and take ϕ to be trace-free. The stability condition is that

$$(1.4) \quad \frac{c_1(F) \cdot h}{\text{rank}(F)} < \frac{c_1(E) \cdot h}{\text{rank}(E)}$$

for all proper ϕ -invariant coherent subsheaves $F \subset E$. Replacing $<$ by \leq defines semistability, while polystable pairs are direct sums of stable Higgs pairs of the same slope. The closely related but slightly more refined notion of Gieseker stability is described in Section 2.1.

Thus the moduli space of solutions of the Vafa-Witten equations has an obvious partial compactification given by taking (semi)stable (hence torsion-free) rank $r > 0$ Higgs *sheaves*. In this paper (but *not* in paper II [TT2]) we always assume the Chern classes of E are chosen so that *semistability and stability coincide*.

1.4. Spectral construction. Finally, we can turn such pairs (E, ϕ) into compactly supported *stable torsion sheaves* \mathcal{E}_ϕ on the Calabi-Yau 3-fold X , where X is the total space of the canonical line bundle K_S of S .

Roughly speaking, over each point $x \in S$, we replace (E_x, ϕ_x) by the eigenspaces of $\phi_x \in \text{Hom}(E_x, E_x \otimes (K_S)_x)$ supported on their respective eigenvalues in $(K_S)_x$. For more details, see Section 2.

1.5. Localisation and a $U(r)$ Vafa-Witten invariant. After fixing Chern classes (r, c_1, c_2) on S for which semistability = stability, we let \mathcal{N} denote the moduli space of stable Higgs sheaves (E, ϕ) on S , or, equivalently, compactly supported stable torsion sheaves \mathcal{E} on X .

As a moduli space of sheaves on Calabi-Yau 3-fold, the results of [HT, Th] give a perfect obstruction theory on \mathcal{N} . Since \mathcal{N} is noncompact, the resulting virtual cycle is uninteresting. However, it also carries a \mathbb{C}^* action, given by scaling the Higgs field ϕ , or, equivalently, the fibres of $X = K_S \rightarrow S$. The \mathbb{C}^* -fixed locus is compact, so we may apply virtual localisation [GP] to define a numerical invariant *counting* the sheaves \mathcal{E} . This is our *preliminary* $U(r)$ Vafa-Witten invariant

$$(1.5) \quad \widetilde{\text{VW}}_{r, c_1, c_2}(S) = \int_{[\mathcal{N}_{r, c_1, c_2}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \in \mathbb{Q},$$

described in Section 4. It is nothing but a local surface DT invariant of the Calabi-Yau 3-fold $X = K_S$.

1.6. Virtual technicalities and an $SU(r)$ Vafa-Witten invariant.

However, as discussed in Remark 4.5, $\widetilde{\text{VW}}$ (1.5) is *zero* unless both $H^{0,1}(S) = 0 = H^{0,2}(S)$. Most surfaces satisfying this condition satisfy a vanishing theorem, making the Vafa-Witten invariant just the (signed) topological Euler characteristic of the moduli space of stable sheaves on S . We would like to be able to handle more interesting cases.

The right fix is to consider the Vafa-Witten equations for the gauge group $SU(r)$ instead of $U(r)$. Thus we restrict to Higgs pairs (E, ϕ) where E has *fixed determinant*¹ and ϕ is *trace-free*. Equivalently, we work with torsion sheaves \mathcal{E} on X whose “centre of mass” (the sum of its points of support in K_S) on each fibre is zero, and whose pushdown to S has fixed determinant.

Producing the right deformation theory for such things turned out to be unexpectedly complicated. At a single point $(E, \phi) \in \mathcal{N}$, the complication is roughly the following.

A natural resolution of the torsion sheaf \mathcal{E}_ϕ on $X \xrightarrow{\pi} S$ in terms of π^*E and $\pi^*\phi$ (2.12) gives a long exact sequence

$$(1.6) \quad \cdots \longrightarrow \text{Hom}(E, E \otimes K_S) \longrightarrow \text{Ext}^1(\mathcal{E}_\phi, \mathcal{E}_\phi) \longrightarrow \text{Ext}^1(E, E) \longrightarrow \cdots$$

relating the automorphisms, deformations and obstructions of \mathcal{E}_ϕ to those of (E, ϕ) . The third arrow is easily seen to take deformations of \mathcal{E}_ϕ to the corresponding deformations of its pushdown $E = \pi_*\mathcal{E}_\phi$. What was surprisingly hard (for us) to show is that the second arrow is the one would expect — namely the derivative of the map from (an open set in) $\text{Hom}(E, E \otimes K_S)$ to \mathcal{N} that takes ϕ to (E, ϕ) .

¹ $SU(r)$ is a bit of a misnomer since for this definition we allow ourselves to fix $\det E = L$ for some *nontrivial* line bundle L on S (e.g. one whose degree is coprime to $\text{rank}(E)$ so that semistability implies stability.) In paper II [TT2] we use fixed *trivial* $\det E = \mathcal{O}_S$.

Dealing with this issue (in full generality, over an arbitrary family to all orders at the level of perfect obstruction theories) is what takes up all of Section 5.

The result is that we can express our perfect obstruction theory for \mathcal{N} in terms of Higgs data on S instead of sheaves on X . That done, it allows us to easily deduce a *symmetric perfect obstruction theory* for the moduli space

$$\mathcal{N}_L^\perp = \{(E, \phi) : \det E \cong L, \operatorname{tr} \phi = 0\}$$

of *trace-free* Higgs fields on sheaves with *fixed determinant* $L \in \operatorname{Pic}(S)$. In fact, using the trace and identity maps on S on the first and last terms of (1.6) produces a splitting

$$(1.7) \quad \operatorname{Ext}^*(\mathcal{E}, \mathcal{E}) \cong H^{*-1}(K_S) \oplus H^*(\mathcal{O}_S) \oplus \operatorname{Ext}^*(\mathcal{E}, \mathcal{E})_\perp.$$

While the first term governs the deformations and obstructions of $\mathcal{E} \in \mathcal{N}$, the last term Ext_\perp governs the same for \mathcal{N}_L^\perp ; see Theorem 6.1.

Thus we get a better definition of the Vafa-Witten invariant by localising this new virtual cycle,

$$(1.8) \quad \operatorname{VW}_{r, c_1, c_2} := \int_{[(\mathcal{N}_{r, L, c_2}^\perp)^{\mathbb{C}^*}]^{\operatorname{vir}}} \frac{1}{e(N^{\operatorname{vir}})} \in \mathbb{Q}.$$

If the Higgs field vanishes $\phi = 0$ for all points $(E, \phi) \in (\mathcal{N}_L^\perp)^{\mathbb{C}^*}$ then (1.8) is actually an integer (the virtual signed Euler characteristic of Section 7), but in general we work in the localised equivariant cohomology $H_{\mathbb{C}^*}^*(BC^*, \mathbb{Q})[t^{-1}] \cong \mathbb{Q}[t, t^{-1}]$ and get rational numbers. (The invariant is a constant, rather than a more general Laurent polynomial in the equivariant parameter t , because the virtual dimension of the problem is zero.)

$\operatorname{VW}_{r, c_1, c_2}$ (1.8) is invariant under deformations of L , and deformations of S which keep the class $c_1 := c_1(L) \in H^2(S)$ of Hodge type (1,1). Physics seems to predict that it *ought* to also be invariant under deformations of the polarisation $\mathcal{O}_S(1)$ (or more generally the stability condition defining \mathcal{N}) — i.e. the invariants' wall crossing should be trivial. We intend to return to this point later using an extension of the wall crossing formula [KL2]. (Whereas the wall crossing formulae of Joyce-Song [JS] and Kontsevich-Soibelman [KS] use weighted Euler characteristics in a crucial way, Kiem-Li use virtual localisation.) When $L = \mathcal{O}_S$ the closely related invariant vw defined by Behrend localisation in paper II [TT2] *does* have trivial wall crossing.

The integral (1.8) is over the different components of the fixed locus of the \mathbb{C}^* action scaling the Higgs field. They come in two flavours.

(1.9) $\phi = 0$. Here we recover $\mathcal{M}_{\operatorname{asd}}$ or the moduli space \mathcal{M}_L of stable sheaves of fixed determinant L on S . By Proposition 7.4 the contribution of this locus to our invariant is the virtual signed Euler characteristic of $\mathcal{M}_{\operatorname{asd}}$ studied in [JT, GK], generalising (1.11). It is an integer.

(1.10) $\phi \neq 0$. We call the union of these components of the fixed locus \mathcal{M}_2 . They can be described in terms of nested Hilbert schemes of

S ; see [GSY1, GSY2] and Section 8. They contribute new rational numbers to the Vafa-Witten invariants.

1.7. A quicker route: derived algebraic geometry. This perfect obstruction theory for Higgs sheaves is of independent interest, but adds a great deal of complexity and length to the paper. It also requires us to work with Illusie’s full cotangent complex² [Ill] instead of the truncated cotangent complex that normally suffices [HT].

Most readers should therefore ignore most of Sections 3, 4, 5 and skip straight to Section 6, taking on trust that the standard perfect obstruction theory [HT, Th] can be refined to the fixed trace and determinant case. But readers who are both derived and stacky will notice that a much quicker solution is to use the results of [STV, TVa] and the theory of derived algebraic stacks.

Namely, one shows that the stack \mathcal{M}_L of rank r torsion-free coherent sheaves E on S with fixed-determinant $\det E \cong L$ has a natural derived structure, inducing one on its (-1) -shifted cotangent bundle³ of $T^*[-1]\mathcal{M}_L$. This is the moduli stack of all Higgs pairs. The open substack of *stable* Higgs pairs is then the product of a derived scheme \mathcal{N}_L^\perp and $T^*[-1]BC^*$:

$$\mathcal{N}_L^\perp \times T^*[-1]BC^* \subset T^*[-1]\mathcal{M}_L.$$

Rigidifying (removing the $T^*[-1]BC^*$ factor) gives a natural derived structure on \mathcal{N}_L^\perp which is *quasi-smooth*. It therefore gives rise to a perfect obstruction theory on the underlying scheme. (This obstruction theory is also *symmetric* [Ca].) We are the wrong authors to use this technology honestly or competently, however, so we employ more classical (but lengthy!) techniques.

1.8. Cotangent field theories. There is a symmetry between the domain and target of the VW equations (1.1). If we add a local Coulomb gauge fixing equation $d_{A_0}^*(A - A_0) \in \Omega^0(\mathfrak{g}_P)$ instead of dividing by gauge, and notice that \mathcal{A}_P is an affine space modelled on $\Omega^1(\mathfrak{g}_P)$, we see both sides contain the same pieces.

Roughly speaking, the equations come from the anti-self-dual (asd) equations $F_A^+ = 0$ by passing to their “ (-1) -shifted cotangent bundle”, at least to first order in the new (cotangent) variables B, Γ .

²Because the exact triangle relating the cotangent complexes of \mathcal{N} , \mathcal{M} and \mathcal{N}/\mathcal{M} need not be exact after truncating.

³At the level of points, this fibres over \mathcal{M}_L with fibre over E the *dual* of the obstruction space $\mathrm{Ext}^2(E, E)_0$ at that point. By the Serre duality $\mathrm{Ext}^2(E, E)_0^* \cong \mathrm{Hom}(E, E \otimes K_S)_0$ we indeed recover (1.3).

The local model is the following. Start with a section s of a vector bundle E over an ambient space A cutting out a moduli space of solutions \mathcal{M} :

$$s^{-1}(0) = \mathcal{M} \subset A. \quad \begin{array}{c} E \\ \downarrow \wr s \end{array}$$

The setting could be real or holomorphic, finite or infinite dimensional. In the latter case it is called a *balanced*, *co-* or *cotangent* field theory in different references; see for example [CMR, DiM, Co, JT] and [VW, Section 2]. For the real VW equations⁴ we start with the asd equations, so the section is F_A^+ in the infinite dimensional bundle $\Omega^+(\mathfrak{g}_P)$ over the infinite dimensional ambient space \mathcal{A}_P , plus the Coulomb gauge fixing equation (or we replace \mathcal{A}_P by its quotient by gauge). Later we will work on a projective surface with a *finite dimensional holomorphic* version of this model, but (we will see below) we have to take A to be a stack rather than a space.

Next pass to the bigger ambient space: the total space of $E^* \rightarrow A$ (in the VW setting this adds the variables B and Γ). By pairing the section s of E with points of E^* it defines a function \tilde{s} on this ambient space E^* . Its gradient $d\tilde{s}$ is a section of the (co)tangent bundle of E^* .

$$(d\tilde{s})^{-1}(0) \subset E^*. \quad \begin{array}{c} T_{E^*}^* \\ \downarrow \wr d\tilde{s} \end{array}$$

The zeros of $d\tilde{s}$ include the original zeros $s = 0$ (inside the zero section A of E^* — i.e. it includes the asd moduli space where $F_A^+ = 0 = B = \Gamma$), but in general there are more, giving the (dual) obstruction bundle on the asd moduli space (those (B, Γ) perpendicular to the image of the linearised asd equation).

In “good” cases⁵ [VW, pp 23–25] the only solutions have $F_A^+ = 0 = B = \Gamma$. In this case the original asd moduli space \mathcal{M}_{asd} is smooth with zero obstruction bundle. Thought of as the moduli space of solutions to the Vafa-Witten equations modulo gauge, however, it has obstruction bundle the (co)tangent bundle of \mathcal{M}_{asd} , and so a natural associated integer invariant given by its Euler class

$$(1.11) \quad \pm e(\mathcal{M}_{\text{asd}}).$$

When no vanishing result holds, it seems no one has yet managed to make a rigorous definition of the VW invariant on a general 4-manifold.

⁴In fact their linearisation about $B = 0 = \Gamma$.

⁵There are different situations in which one can prove such a vanishing theorem, but they all involve S having curvature which is positive in some sense.

1.9. The complex case is more linear. Vafa and Witten's equation is a nonlinear version of the above construction including also the quadratic term $[B.B] + [B, \Gamma]$. But in the projective surface case, the Hitchin-Kobayashi correspondence *removes this nonlinearity at the expense of enlarging \mathcal{M}_{asd}* .

That is, if we let \mathcal{M} denote the moduli *stack* of *all* bundles (or torsion-free sheaves) on S — not just those which are stable, which form \mathcal{M}_{asd} — then the stable Higgs sheaves form a Zariski open in the (-1) -shifted cotangent bundle

$$\mathcal{N} \subset T^*[-1]\mathcal{M}.$$

As a space (ignoring its derived structure or obstruction theory) $T^*[-1]\mathcal{M} = \text{Spec Sym}^\bullet \text{Ob}$, where Ob is the obstruction sheaf of \mathcal{M} , so it can be thought of as the total space of the dual obstruction bundle over \mathcal{M} . Since the fibre of Ob over $E \in \mathcal{M}$ is $\text{Ext}^2(E, E)$, Serre dual to $\text{Hom}(E, E \otimes K_S)$, we see $T^*[-1]\mathcal{M} \rightarrow \mathcal{M}$ is a fibration by Higgs fields $\phi \in \text{Hom}(E, E \otimes K_S)$.

Passing to the open locus which satisfies the stability condition (1.4) the stabiliser groups drop to \mathbb{C}^* . Rigidifying (removing them, and their (-1) -shifted cotangents in degree -2) gives the scheme \mathcal{N} with a quasi-smooth derived structure. This induces the symmetric obstruction theory we used to define our invariant. Similarly our $SU(r)$ moduli space is an open set

$$\mathcal{N}_L^\perp \subset T^*[-1]\mathcal{M}_L,$$

inducing the correct symmetric obstruction theory, again after rigidifying.

This enlargement of \mathcal{M}_{asd} to \mathcal{M}_L is what results in there now being *two* types of fixed locus for the additional \mathbb{C}^* action that scales the Higgs field ϕ . The first (1.9) is \mathcal{M}_{asd} but the second \mathcal{M}_2 (1.10) with $\phi \neq 0$ is not in $T^*[-1]\mathcal{M}_{\text{asd}}$.

1.10. Other localisations. There are three other natural ways to localise the virtual cycle of \mathcal{N} to $\mathcal{N}^{\mathbb{C}^*}$, thus giving competing definitions of the VW invariant.

Let \mathcal{N} be a scheme with a symmetric perfect obstruction theory. Behrend [Be] defines a constructible function

$$\chi^B: \mathcal{N} \longrightarrow \mathbb{Z}$$

such that if \mathcal{N} is compact then the degree of its (zero dimensional) virtual cycle equals the Euler characteristic of \mathcal{N} weighted by χ^B ,

$$\int_{[\mathcal{N}]^{\text{vir}}} 1 = e(\mathcal{N}, \chi^B) := \sum_{i \in \mathbb{Z}} i \cdot e((\chi^B)^{-1}(i)).$$

Since our moduli spaces of Higgs pairs \mathcal{N} and \mathcal{N}_L^\perp are noncompact, we instead take the right hand side as a definition. Of course χ^B is \mathbb{C}^* -invariant and the Euler characteristic of any non-fixed orbit is 0, so only the fixed points contribute. This gives the localisation

$$(1.12) \quad e(\mathcal{N}, \chi^B) = e(\mathcal{N}^{\mathbb{C}^*}, \chi^B|_{\mathcal{N}^{\mathbb{C}^*}}).$$

When $h^{0,1}(S) > 0$ this vanishes (since $e(\text{Jac}(S)) = 0$) so we fix determinant and trace and localise on \mathcal{N}_L^\perp instead. Thus we consider the Kai-localised $SU(r)$ Vafa-Witten invariant given by⁶

$$(1.13) \quad \mathbf{vw}_{r,c_1,c_2} := e\left((\mathcal{N}_L^\perp)^{\mathbb{C}^*}, \chi^B|_{(\mathcal{N}_L^\perp)^{\mathbb{C}^*}}\right) \in \mathbb{Z}.$$

When $h^{0,1}(S) = 0$ we recover (1.12). We study \mathbf{vw} in paper II [TT2].

Similarly one can also use Kiem-Li's cosection localisation [KL1]. Differentiating the \mathbb{C}^* action on \mathcal{N} and \mathcal{N}_L^\perp defines vector fields thereon. Since these have symmetric obstruction theories, their obstruction sheaves are their cotangent sheaves. Pairing with the vector field gives *cosections*

$$\text{Ob}_{\mathcal{N}} \longrightarrow \mathcal{O}_{\mathcal{N}} \quad \text{and} \quad \text{Ob}_{\mathcal{N}_L^\perp} \longrightarrow \mathcal{O}_{\mathcal{N}_L^\perp}.$$

Their zero loci are the \mathbb{C}^* -fixed loci, so Kiem-Li's construction gives localised cycles

$$[\mathcal{N}]^{\text{loc}} \in A_0(\mathcal{N}^{\mathbb{C}^*}) \quad \text{and} \quad [\mathcal{N}_L^\perp]^{\text{loc}} \in A_0((\mathcal{N}_L^\perp)^{\mathbb{C}^*}).$$

When $h^{0,1}(S) = 0$ they coincide, otherwise the first vanishes. This suggests a third VW invariant

$$\mathbf{vw}'_{r,c_1,c_2} := \int_{[\mathcal{N}_L^\perp]^{\text{loc}}} 1.$$

Thirdly, we could take the signed topological Euler characteristic of $(\mathcal{N}_L^\perp)^{\mathbb{C}^*}$. In fact these three rival definitions are closely related. By [Ji, JT] the first two are equal,

$$\mathbf{vw} = \mathbf{vw}',$$

but are different in general from the invariant \mathbf{VW} defined by virtual localisation. Furthermore, when a “vanishing theorem” holds so that all \mathbb{C}^* -fixed stable Higgs pairs have $\phi = 0$ (equivalently \mathbb{C}^* -fixed stable torsion sheaves on X are all pushed forward from S) they also equal the third definition by “dimension reduction” [BBS, Da], [JT, Section 5]:

$$\mathbf{vw} = \mathbf{vw}' = (-1)^{\text{vd}} e\left((\mathcal{N}_L^\perp)^{\mathbb{C}^*}\right).$$

Furthermore we expect that $\mathbf{vw} = \mathbf{VW}$ for surfaces with $K_S \leq 0$ [TT2].

More generally \mathbf{vw} has the advantages that it is often more easily computable and admits a natural refinement and categorification (as expected in physics). But most importantly, its disadvantages are that it is not (in general) deformation invariant, and physics prefers \mathbf{VW} over \mathbf{vw} for modularity reasons that we describe next.

⁶At least when there are no strictly semistables. More generally we use the Joyce-Song formalism and consider invariants in \mathbb{Q} counting semistables [TT2].

1.11. Calculations and modularity. Vafa-Witten predicted that for fixed rank r and determinant L , S-duality should force the generating function

$$Z_r(S, L) := q^{-s} \sum_{n \in \mathbb{Z}} \mathbf{VW}_{r, c_1(L), n}(S) q^n$$

to be a modular form for some appropriate shift s . In fact their examples indicated that one should take $s = e(S)/12$, and that the result is a modular form of weight $w/2 = -e(S)/2$.

There have been many compelling calculations on the first fixed component (1.9) confirming these predictions, usually in the presence of a vanishing theorem (for instance when $K_S \leq 0$) ensuring that the second component \mathcal{M}_2 (1.10) is empty. Recent work of Göttsche and Kool [GK] extends these calculations to general type surfaces S .

However, the fixed components of the second kind (1.10) with nonzero Higgs field seem to have been ignored until now. In Section 8 we compute the part of the generating series of Vafa-Witten invariants \mathbf{VW} (1.8) coming from a series of components with nonzero Higgs field. With some tricks we manage to sum the series in closed form in Proposition 8.23. The result is far from modular — in fact it is an *algebraic* function of q ,

$$(1.14) \quad c(1-q)^{g-1} \left(1 + \frac{1-3q}{\sqrt{(1-q)(1-9q)}} \right)^{1-g},$$

where $c = (-1)^{p_g(S)+g} \cdot 2^{-p_g(S)-1}$ and $g-1 = c_1(S)^2$.

In [TT2] we repeat the calculation for the Kai localised invariant \mathbf{vw} (1.13), and find something which *is* modular form up to a factor of $(1-q)^{e(S)}$. This misled us for some time, but it turns out that it is **not** an indication that the Vafa-Witten invariant \mathbf{vw} (1.13) is the one relevant to physics. In fact Martijn Kool pointed out that the coefficients of q^0 and q^1 in (1.14) match those in a modular form in [VW, Equation 5.38]. So in the rest of Section 8 we compute the contributions of more components of \mathcal{M}_2 (1.10) to \mathbf{VW} . When added to (1.14) we get agreement also in q^2 and q^3 . That is, we find in (8.38) that

$$\begin{aligned} 2^{p_g(S)+g} \sum_{n \in \mathbb{Z}} \mathbf{VW}_{2, K_S, n}(S) q^n &= \\ \prod_{n=1}^{\infty} (1 - q^{2n})^{-12(p_g(S)+1)} (1 - q^n)^{2g-2} \left(\sum_{j=0}^{\infty} q^{j^2+j} \right)^{1-g} & \pmod{q^4}. \end{aligned}$$

The right hand side is the (modular) second term in [VW, Equation 5.38].

This may not sound like much, but given the complexity of the calculations across 15 pages, we feel it cannot be a coincidence. The upshot is that we now believe the virtual localisation invariant \mathbf{VW} of this paper to be the “correct” one for physics.⁷ Remarkably, it seems that the generating series

⁷At least in the stable case; for semistable Higgs pairs see [TT2].

of invariants coming from the two types of fixed component (1.9, 1.10) are *separately modular*, but the contributions of natural series of subcomponents of \mathcal{M}_2 (1.10) are not by themselves modular.

Even more remarkably, Vafa and Witten predicted the above formula without calculating on the nested Hilbert schemes making up \mathcal{M}_2 (1.10). It seems our geometric techniques are still a lot less powerful than the physics of the early 1990s. The Kiem-Li cosection localisation we use in Section 8.2 may be a primitive analogue of their perturbation [VW, Equation 5.19] which they use to localise their calculation to “cosmic string” contributions. Our results seem to indicate that the Gieseker compactification of moduli of sheaves is relevant to physics (which usually uses the Uhlenbeck compactification).

The alternative definition \mathbf{vw} (1.13) has its own modularity properties [TT2], but these are the *wrong* ones for physics, except, it seems, when $K_S \leq 0$. We conjecture in [TT2] that $\mathbf{VW} = \mathbf{vw}$ when $K_S \leq 0$ (we prove it for $\deg K_S < 0$ and sketch some reasoning for the belief when $K_S = \mathcal{O}_S$). We also perform calculations with \mathbf{vw} on K3 surfaces that recover and generalise more modular forms predicted in [VW].

1.12. Semistable case. The great advantage of the “wrong” definition \mathbf{vw} (1.13) is that, using the Behrend function and weighted Euler characteristics, one can use the Joycian (or Kontsevich-Soibelman) formalism to define (rational) invariants when there are strictly semistable Higgs pairs.

In the second paper [TT2] we use this theory, especially for K3 surfaces. We then conjecture a similar formalism can be used in the virtual localisation setting to extend \mathbf{VW} to the semistable case too.

That is, motivated by Mochizuki’s work [Mo] and Joyce-Song pairs [JS] we virtually enumerate certain stable pairs

$$(\mathcal{E}_\phi, s)$$

on $X = K_S$ (or equivalently triples (E, ϕ, s) on S .) Here the torsion sheaf \mathcal{E}_ϕ has centre of mass zero on the K_S fibres (equivalently $\mathrm{tr} \phi = 0$), and $\det \pi_* \mathcal{E}_\phi = \det E \cong \mathcal{O}_S$. Finally

$$s \in H^0(X, \mathcal{E}_\phi(N)) \cong H^0(S, E(N))$$

for some fixed $N \gg 0$. There is a symmetric obstruction theory for such pairs, given by combining the $R\mathrm{Hom}_\perp$ perfect obstruction theory of (1.7) with Joyce-Song’s pairs theory. We conjecture the resulting invariants can be written in terms of universal formulae in N with coefficients given by Vafa-Witten invariants. These recover the invariants \mathbf{VW} (1.8) when stability and semistability coincide, and seem to give the “right” answer in other calculations.

1.13. Kapustin-Witten equations on projective surfaces. It would be natural to repeat the same trick for the Kapustin-Witten equations [KW] on a projective surface S . On any Kähler surface the equations reduce [Ta2] to

Simpson's equations, for which there is a Hitchin-Kobayashi correspondence [Si]. Via the spectral construction, solutions can be identified with torsion sheaves on the Calabi-Yau 4-fold

$$X = T^*S.$$

Borisov-Joyce [BJ] describe a *real* analogue of perfect obstruction theory and virtual cycle for sheaves on Calabi-Yau 4-folds X , and Cao-Leung [CL] conjecture a way to extend virtual localisation to this setting. Applying this to the natural \mathbb{C}^* action on the fibres of $X \rightarrow S$ (which does not preserve the Calabi-Yau form) one can define Kapustin-Witten invariants for S . We intend to return to this in future work.

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1.14. Notation. To keep the notation readable we suppress some obvious pullback maps, and, given a map, we often use the same notation for any basechange thereof. So for instance π will denote the standard projection $X \rightarrow S$ of (2.1), but also the induced map $\mathcal{N} \times X \rightarrow \mathcal{N} \times S$ (which should properly be called $\text{id}_{\mathcal{N}} \times \pi$).

Therefore when we let p_S denote the projection $S \rightarrow \text{Spec } \mathbb{C}$ it also denotes the projections $\mathcal{N} \times S \rightarrow \mathcal{N}$ and $\mathcal{M} \times S \rightarrow \mathcal{M}$.

Where possible we stick to the notation of

$$(E, \phi) \quad \text{and} \quad \mathcal{E} = \mathcal{E}_\phi$$

for a Higgs pair on S and the associated torsion sheaf on $X = K_S$ respectively. In families these get replaced by

$$(\mathbf{E}, \Phi) \quad \text{and} \quad \mathcal{E} = \mathcal{E}_\Phi$$

respectively. For instance these could denote the universal Higgs pair over $\mathcal{N} \times S$ and the universal torsion sheaf over $\mathcal{N} \times X$ respectively.

We reserve \mathcal{M} for moduli spaces of (semi)stable sheaves on S , while \mathcal{M} denotes the moduli *stack* of all coherent sheaves on S . Similarly \mathcal{N} is used for moduli spaces of (semi)stable Higgs pairs on S or, equivalently, compactly supported torsion sheaves on $X = K_S$.

Given a map π , we denote by $\mathcal{H}om_\pi$ the functor $\pi \circ \mathcal{H}om$ of Homs down the fibres of π , and by $R\mathcal{H}om_\pi \cong R\pi_* R\mathcal{H}om$ its derived functor.

For a sheaf or complex of sheaves \mathcal{F} , we use $\mathcal{F}^\vee = R\mathcal{H}om(\mathcal{F}, \mathcal{O})$ to denote its derived dual. For a single sheaf, \mathcal{F}^* denotes its underived dual $h^0(\mathcal{F}^\vee)$.

We use \mathbb{L} to denote Illusie's cotangent complex, and $\mathbb{T} := \mathbb{L}^\vee$ for its dual, the tangent complex.

2. SPECTRAL CONSTRUCTION

Fix a complex projective algebraic surface S and let

$$(2.1) \quad \pi: X \longrightarrow S$$

be the total space of a line bundle $L \rightarrow S$. (We will mainly be interested in the case $L = K_S$.) We recall the classical spectral construction relating L -Higgs pairs (E, ϕ) on S to compactly supported sheaves \mathcal{E}_ϕ on X .

Roughly speaking, over each point $x \in S$, we replace (E_x, ϕ_x) by the eigenspaces of $\phi_x \in \text{Hom}(E_x, E_x \otimes L_x)$ supported on their respective eigenvalues in L_x . More precisely we get an equivalence of categories as follows.

Proposition 2.2. *There is an abelian category $\text{Higgs}_L(S)$ of L -Higgs pairs (E, ϕ) on S , and an equivalence of categories*

$$(2.3) \quad \text{Higgs}_L(S) \cong \text{Coh}_c(X)$$

with the category of compactly supported coherent sheaves on X .

Proof. The maps in $\text{Higgs}_L(S)$ between (E, ϕ) and (F, ψ) are maps $f: E \rightarrow F$ inducing a commutative diagram

$$(2.4) \quad \begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes L \\ f \downarrow & & f \otimes \text{id}_L \downarrow \\ F & \xrightarrow{\psi} & F \otimes L. \end{array}$$

Taking kernels and cokernels of the columns defines kernel and cokernel Higgs pairs.

It is a classical result of Serre (see for instance [Ha, Exercise II.5.17]) that for affine maps $\pi: X \rightarrow S$, the functor π_* is an equivalence between the abelian category of coherent \mathcal{O}_X -modules and the abelian category of $\pi_* \mathcal{O}_X$ -modules on S . In our case

$$(2.5) \quad \pi_* \mathcal{O}_X = \bigoplus_{i \geq 0} L^{-i} \cdot \eta^i,$$

where η is the tautological section of $\pi^* L$ which is linear on the fibres and cuts out the zero section $S \subset X$. The affine X/S is Spec of this sheaf of \mathcal{O}_S -algebras, and sheaves \mathcal{E} on X are equivalent to sheaves of modules $\pi_* \mathcal{E}$ over $\pi_* \mathcal{O}_X$ (2.5).

But (2.5) is generated by \mathcal{O}_S and $L^{-1} \cdot \eta$, so a module over it is equivalent to an \mathcal{O}_S -module E together with a commuting action of $L^{-1} \cdot \eta$, i.e. an \mathcal{O}_S -linear map

$$E \otimes L^{-1} \xrightarrow{\pi_* \eta} E.$$

Thus we get an L -Higgs pair

$$(2.6) \quad (E, \phi) = (\pi_* \mathcal{E}, \pi_* \eta).$$

Conversely, given a Higgs pair (E, ϕ) we get an action of $L^{-i} \cdot \eta^i$ by

$$(2.7) \quad E \otimes L^{-i} \xrightarrow{\phi^i} E.$$

Summing over i gives an action of $\pi_* \mathcal{O}_X$ (2.5) on E . We denote by \mathcal{E}_ϕ the sheaf on X that this $\pi_* \mathcal{O}_X$ -module defines.

This equivalence is of course a functor: morphisms of sheaves \mathcal{E} induce morphisms of their pushdowns $E = \pi_* \mathcal{E}$ which commute with the action of η , therefore giving commutative diagrams (2.4) and so morphisms in $\text{Higgs}_L(S)$.

Finally, for \mathcal{E} coherent, the quasicoherent sheaf $\pi_* \mathcal{E}$ is coherent if and only if $\pi|_{\text{supp } \mathcal{E}}$ is proper, if and only if \mathcal{E} is compactly supported. \square

2.1. Gieseker stability. The L -Higgs pair (E, ϕ) is said to be *Gieseker stable* with respect to an ample line bundle $\mathcal{O}_S(1)$ if and only if

$$(2.8) \quad \frac{\chi(F(n))}{\text{rank}(F)} < \frac{\chi(E(n))}{\text{rank}(E)} \quad \text{for } n \gg 0,$$

for every proper ϕ -invariant subsheaf $F \subset E$ on S . Here χ denotes holomorphic Euler characteristic. Replacing $<$ by \leq defines semistability.

Gieseker stability implies slope semistability of (E, ϕ) , and is implied by slope stability. Recall (1.4) this is defined by the inequality

$$\frac{c_1(F) \cdot h}{\text{rank}(F)} < \frac{c_1(E) \cdot h}{\text{rank}(E)}, \quad h := c_1(\mathcal{O}(1)).$$

When the denominator and numerator of the right hand side (the rank and degree of E) are coprime, we further find that

$$\text{slope semistable} = \text{Gieseker semistable} = \text{Gieseker stable} = \text{slope stable}.$$

Lemma 2.9. *Under the equivalence of categories (2.3), Gieseker (semi)-stability of the L -Higgs pair (E, ϕ) with respect to $\mathcal{O}_S(1)$ is equivalent to Gieseker stability of the sheaf \mathcal{E}_ϕ with respect to $\pi^* \mathcal{O}_S(1)$.*

Proof. Under the equivalence, ϕ -invariant subsheaves $F \subset E$ are equivalent to subsheaves $\mathcal{F} \subset \mathcal{E}_\phi$ on X . Moreover $\chi(E) = \chi(\pi_* \mathcal{E}_\phi) = \chi(\mathcal{E}_\phi)$, so Gieseker stability (2.8) of (E, ϕ) is equivalent to

$$(2.10) \quad \frac{\chi(\mathcal{F}(n))}{r(\mathcal{F})} < \frac{\chi(\mathcal{E}_\phi(n))}{r(\mathcal{E}_\phi)} \quad \text{for } n \gg 0,$$

for all proper subsheaves $\mathcal{F} \subset \mathcal{E}_\phi$. Here we have suppressed the pullback map π^* on $\mathcal{O}(1)$, and denoted by $r(\mathcal{E}_\phi) := \int_X c_1(\mathcal{E}_\phi) \cdot h^2 = \text{rank}(E) \int_S h^2$ the leading coefficient of its Hilbert polynomial $\chi(\mathcal{E}_\phi(n))$.

But (2.10) is the usual Gieseker stability for the torsion sheaf \mathcal{E}_ϕ on X , so these two notions of Gieseker stability for (E, ϕ) and \mathcal{E}_ϕ coincide. \square

2.2. Resolution. Since \mathcal{E}_ϕ is generated by its sections down π we have a natural surjection $\pi^*E = \pi^*\pi_*\mathcal{E} \xrightarrow{\text{ev}} \mathcal{E}_\phi$. Its kernel is given by the following.⁸

Proposition 2.11. *There is an exact sequence*

$$(2.12) \quad 0 \longrightarrow \pi^*(E \otimes L^{-1}) \xrightarrow{\pi^*\phi - \eta} \pi^*E \xrightarrow{\text{ev}} \mathcal{E}_\phi \longrightarrow 0.$$

Proof. Consider $X \times_S X$ — the total space of $L^{\oplus 2}$ over S — with its two natural projections π_1, π_2 to X and projection p to S . On it is an obvious exact sequence

$$(2.13) \quad 0 \longrightarrow \pi_1^*\mathcal{E}(-\Delta) \longrightarrow \pi_1^*\mathcal{E} \longrightarrow \Delta_*\mathcal{E} \longrightarrow 0,$$

where $\Delta \cong X \subset X \times_S X$ is the diagonal.

Now p^*L has two tautological sections η_1, η_2 whose difference $\eta_1 - \eta_2$ cuts out the diagonal. Thus $p^*L \cong \mathcal{O}(\Delta)$ and we can rewrite (2.13) as

$$0 \longrightarrow \pi_1^*\mathcal{E} \otimes \pi_2^*\pi^*L^{-1} \xrightarrow{\eta_1 - \eta_2} \pi_1^*\mathcal{E} \longrightarrow \Delta_*\mathcal{E} \longrightarrow 0.$$

Now apply π_{2*} , using the projection formula and the flat basechange formula $\pi_{2*}\pi_1^* = \pi^*\pi_*$. By (2.6) we get (2.12). \square

2.3. Deformation theory. This resolution (2.12) of \mathcal{E} is useful to relate the deformation theory of \mathcal{E} , governed by

$$\text{Ext}^*(\mathcal{E}, \mathcal{E}),$$

with that of the Higgs pair (E, ϕ) , governed by the cohomology groups of the total complex of

$$R\text{Hom}(E, E) \xrightarrow{[\cdot, \phi]} R\text{Hom}(E, E \otimes L).$$

Proposition 2.14. *There is an exact triangle*

$$(2.15) \quad R\text{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow R\text{Hom}(E, E) \xrightarrow{\circ\phi - \phi\circ} R\text{Hom}(E \otimes L^{-1}, E).$$

Proof. Applying $R\text{Hom}(\cdot, \mathcal{E})$ to (2.12) gives the exact triangle

$$R\text{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow R\text{Hom}(\pi^*E, \mathcal{E}) \xrightarrow{\circ\pi^*\phi - \eta} R\text{Hom}(\pi^*(E \otimes L^{-1}), \mathcal{E}).$$

⁸The result says that if we divide π^*E by the minimal submodule to ensure that η acts as $\pi^*\phi$ on the quotient, we get \mathcal{E}_ϕ .

By adjunction this equals

$$R\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow R\mathrm{Hom}(E, \pi_* \mathcal{E}) \xrightarrow{\circ\phi - \pi_* \eta} R\mathrm{Hom}(E \otimes L^{-1}, \pi_* \mathcal{E}).$$

Since $\pi_* \eta: \pi_* \mathcal{E} \rightarrow \pi_* \mathcal{E} \otimes L$ is $\phi: E \rightarrow E \otimes L$ by (2.6) this is

$$R\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow R\mathrm{Hom}(E, E) \xrightarrow{\circ\phi - \phi \circ} R\mathrm{Hom}(E, E \otimes L). \quad \square$$

In particular, taking cohomology gives the long exact sequence (1.6) of the introduction.

2.4. In families. From now on we fix $L = K_S$. We also work with *families* of sheaves over S and X . To prove deformation invariance of our invariants, we will also want to allow S and X to vary smoothly. So we now let $S \rightarrow B$ be a smooth projective morphism with 2-dimensional fibres, and we let $X \rightarrow B$ denote the total space of its relative canonical bundle $K_{S/B}$. (On a first viewing, the reader should set $B = \mathrm{Spec} \mathbb{C}$ and forget all about it.)

Let $\mathcal{N} \rightarrow B$ denote the moduli space of Gieseker stable Higgs pairs on the fibres of $S \rightarrow B$ with fixed rank⁹ and Chern classes (r, c_1, c_2) . Equivalently it is a moduli space of compactly supported stable torsion sheaves on the fibres of $X \rightarrow B$, with rank 0 and

$$\begin{aligned} (2.16) \quad c_1 &= r[S], \\ c_2 &= -\iota_* \left(c_1 + \frac{r(r+1)}{2} c_1(S) \right), \\ c_3 &= \iota_* \left(c_1^2 - 2c_2 + (r+1)c_1 \cdot c_1(S) + \frac{r(r+1)(r+2)}{6} c_1(S)^2 \right) \end{aligned}$$

in $H_c^*(X_t, \mathbb{Z})$. Here $\iota: S \hookrightarrow X$ is the zero section, $[S]$ its Poincaré dual, and X_t is the fibre of $X \rightarrow B$ over any closed point $t \in B$.

Pick a (twisted)¹⁰ universal sheaf

$$\mathcal{E} \text{ over } \mathcal{N} \times_B X.$$

As usual we let π denote both the projection $X \rightarrow S$ of (2.1) and (cf. Notation section) any basechange such as $\mathcal{N} \times_B X \rightarrow \mathcal{N} \times_B S$. Since \mathcal{E} is flat over \mathcal{N} and π is affine,

$$(2.17) \quad \mathbf{E} := \pi_* \mathcal{E} \text{ over } \mathcal{N} \times_B S$$

is also flat over \mathcal{N} . It is also coherent because \mathcal{E} is finite over $\mathcal{N} \times_B S$. Thus it defines a classifying map

$$(2.18) \quad \begin{aligned} \Pi: \quad \mathcal{N} &\longrightarrow \mathcal{M}, \\ \mathcal{E} &\longmapsto \pi_* \mathcal{E} \quad (\text{or, equivalently, } (E, \phi) \longmapsto E), \end{aligned}$$

⁹In this paper we always take rank $r > 0$. When $r = 0$ the obstruction theory (1.7) has to be modified.

¹⁰Although a global universal sheaf may not exist in general due to \mathbb{C}^* -automorphisms, one can work locally over \mathcal{N} (where one always exists) or globally with a *twisted* universal sheaf. We can ignore this issue since we are only concerned with (derived functors of) $\mathcal{H}om(\mathcal{E}, \mathcal{E})$, which exists globally and glues uniquely and independently of any choices.

to the moduli *stack* \mathcal{M} of coherent sheaves on the fibres of $S \rightarrow B$ of the given rank and Chern classes. We will often not distinguish between the universal sheaf \mathbf{E} over $\mathcal{M} \times S$ and its pullback $\mathbf{E} = \Pi^* \mathbf{E}$ over $\mathcal{N} \times S$.

By the same working as above, over the families

$$p_X: \mathcal{N} \times_B X \rightarrow \mathcal{N} \quad \text{and} \quad p_S: \mathcal{N} \times_B S \rightarrow \mathcal{N}$$

instead of on single copies of X and S , we replace (2.15) by the exact triangle (2.19)

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \xrightarrow{\pi_*} R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E}) \xrightarrow{[\cdot, \phi]} R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S).$$

Later we will see how this relates the obstruction theories of \mathcal{N} , \mathcal{M} and $\Pi: \mathcal{N} \rightarrow \mathcal{M}$. The following will also be useful.

Proposition 2.20. *The exact triangle (2.19) is its own Serre dual.*

Proof. The claim is that replacing each term in (2.19) with its relative Serre dual (down the X - and S -fibres respectively), and the arrows by their duals,

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[3] \leftarrow R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)[2] \leftarrow R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E})[2],$$

we get the same exact triangle, shifted.

For the right hand arrows, this is simple. We concentrate on the left hand arrows. The claim is that the following horizontal arrows

$$\begin{array}{ccc} R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) & \xrightarrow{\pi_*} & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E}) \\ \otimes & & \otimes \\ R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[3] & \xleftarrow{\partial} & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)[2] \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{N}} & & \mathcal{O}_{\mathcal{N}} \end{array}$$

intertwine the vertical pairings given by cup product and trace. Here ∂ is the coboundary morphism of the exact triangle (2.19). That is, identifying (2.21)

$$R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S) \cong R\mathcal{H}om_{p_S}(\mathbf{E} \otimes K_S^{-1}, \pi_* \mathcal{E}) \cong R\mathcal{H}om_{p_X}(\pi^* \mathbf{E} \otimes K_S^{-1}, \mathcal{E}),$$

it is cup product with the canonical extension class

$$(2.22) \quad e \in \text{Ext}_{X \times \mathcal{N}}^1(\mathcal{E}, \pi^* \mathbf{E} \otimes K_S^{-1})$$

of the resolution (2.12). So we want to show that the two parings

$$\begin{array}{ccc} R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \overset{L}{\otimes} R\mathcal{H}om_{p_X}(\pi^* \mathbf{E} \otimes K_S^{-1}, \mathcal{E})[2] & \xrightarrow{\text{tr}_X((\cdot \circ (\cdot) \circ e))} & \\ \parallel & & \\ R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \overset{L}{\otimes} R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)[2] & \xrightarrow{\text{tr}_S(\pi_*(\cdot \circ (\cdot)))} & \mathcal{O}_{\mathcal{N}} \end{array}$$

are equal. Since $\text{tr}(ab) = \text{tr}(ba)$ the upper map may be rewritten

$$(2.23) \quad \text{tr}_S(e \circ (\cdot) \circ (\cdot))$$

when we identify the product of the first two terms in (2.23) as lying in

$$(2.24) \quad R\mathcal{H}om_{p_X}(\mathcal{E}, \pi^*E \otimes K_S^{-1})[1] \cong R\mathcal{H}om_{p_X}(\mathcal{E}, \pi^!E) \cong R\mathcal{H}om_{p_S}(\pi_*\mathcal{E}, E) \cong R\mathcal{H}om_{p_S}(E, E)$$

(since the dualising complex of π is $\pi^*K_S^{-1}[1]$) and the last in

$$R\mathcal{H}om_{p_X}(\pi^*E \otimes K_S^{-1}, \mathcal{E})[2] \cong R\mathcal{H}om_{p_S}(E, E \otimes K_S)[2].$$

The identification (2.24) is isomorphic to the composition

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \pi^!E) \xrightarrow{\pi_*} R\mathcal{H}om_{p_S}(\pi_*\mathcal{E}, \pi_*\pi^!E) \xrightarrow{\epsilon} R\mathcal{H}om_{p_S}(\pi_*\mathcal{E}, E),$$

where $\epsilon: \pi_*\pi^! \rightarrow \text{id}$ is the counit. Therefore the product of the first two terms of (2.23) is

$$\epsilon \circ \pi_*(e \circ (\cdot)) = \epsilon \circ \pi_*(e) \circ \pi_*(\cdot).$$

But $\epsilon \circ \pi_*(e)$ is the image of e under the identification

$$e \in \text{Ext}_{X \times \mathcal{N}}^1(\mathcal{E}, \pi^!E[-1]) \cong \text{Hom}_{S \times \mathcal{N}}(\pi_*\mathcal{E}, E) \cong \text{Hom}_{S \times \mathcal{N}}(E, E),$$

which is id_E . Thus (2.23) is $\text{tr}_S(\pi_*(\cdot) \circ (\cdot))$, as required. \square

Corollary 2.25. *The exact triangle (2.19) fits into the following commutative diagram of exact triangles with split columns*

$$\begin{array}{ccccc} R\mathcal{H}om_{p_S}(E, E \otimes K_S)_0[-1] & \longrightarrow & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_{\perp} & \longrightarrow & R\mathcal{H}om_{p_S}(E, E)_0 \\ \updownarrow & & \updownarrow & & \updownarrow \\ R\mathcal{H}om_{p_S}(E, E \otimes K_S)[-1] & \longrightarrow & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) & \longrightarrow & R\mathcal{H}om_{p_S}(E, E) \\ \text{id} \updownarrow \text{tr} & & \updownarrow & & \text{id} \updownarrow \text{tr} \\ Rp_{S*}K_S[-1] & \longleftrightarrow & Rp_{S*}K_S[-1] \oplus Rp_{S*}\mathcal{O}_S & \longleftrightarrow & Rp_{S*}\mathcal{O}_S, \end{array}$$

where the suffix 0 denotes trace-free Homs. Letting $R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_{\perp}$ denote the co-cone of the central column, it is Serre dual to its own shift by [3].

Proof. The left and right hand columns are split because $\text{tr} \circ \text{id} = \text{rank} \neq 0$.

The composition

$$\mathcal{O}_S \xrightarrow{\text{id}} R\mathcal{H}om_{\pi}(\mathcal{E}, \mathcal{E}) \xrightarrow{\pi_*} R\mathcal{H}om(E, E) \xrightarrow{\text{tr}} \mathcal{O}_S$$

is the identity, so the first arrow is split. Apply Rp_{S*} and use $Rp_{S*}R\pi_* = Rp_{X*}$ to give the maps

$$(2.26) \quad R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \longleftrightarrow Rp_{S*}\mathcal{O}_S$$

lifting the corresponding maps on $R\mathcal{H}om_{p_S}(E, E)$.

The Serre duals of the maps (2.26) give the maps

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \longleftrightarrow Rp_{S*}K_S[-1].$$

By Proposition 2.20 these also commute with the Serre duals of the identity and trace maps on $R\mathcal{H}om_{p_S}(E, E \otimes K_S)[-1]$. Since these are the trace and identity maps respectively, the result follows. \square

This co-cone $R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_{\perp}$ will eventually provide the symmetric perfect obstruction theory we need for the moduli space \mathcal{N}_L^{\perp} of stable trace-free fixed-determinant Higgs pairs; see Theorem 6.1.

3. PERFECT OBSTRUCTION THEORY FOR $U(r)$ THEORY

We will describe the perfect obstruction theories of \mathcal{N} and \mathcal{M} and, for later use, their relationship under Π (2.18). The exact triangle of cotangent complexes

$$\Pi^* \mathbb{L}_{\mathcal{M}/B} \xrightarrow{\Pi^*} \mathbb{L}_{\mathcal{N}/B} \longrightarrow \mathbb{L}_{\mathcal{N}/\mathcal{M}}$$

may fail to be exact if we use truncated cotangent complexes as in [HT]. Therefore we use full cotangent complexes, to which [HT] does not immediately apply. However, since we are deforming sheaves (\mathcal{E} and E) rather than complexes of sheaves, we can use Illusie's seminal work [Ill].

3.1. Atiyah classes. Illusie [Ill, Section III.2.3] defines the Atiyah class of a coherent sheaf \mathcal{F} on a B -scheme Z as follows. Using the notation of Gillam [Gi] we let

$$Z[\mathcal{F}] := \mathrm{Spec}_Z(\mathcal{O}_Z \oplus \mathcal{F}) \xrightarrow{q} Z$$

be the trivial square-zero thickening of Z by \mathcal{F} . Here $\mathcal{O}_Z \oplus \mathcal{F}$ is an \mathcal{O}_Z -algebra in the obvious way: $(f, s) \cdot (g, t) := (fg, ft + gs)$. It carries a \mathbb{C}^* action, fixing \mathcal{O}_Z and acting on \mathcal{F} with weight 1. We get the following exact triangle of \mathbb{C}^* -equivariant cotangent complexes on $Z[\mathcal{F}]$,

$$(3.1) \quad q^* \mathbb{L}_{Z/B} \longrightarrow \mathbb{L}_{Z[\mathcal{F}]/B} \longrightarrow \mathbb{L}_{Z[\mathcal{F}]/Z} \longrightarrow q^* \mathbb{L}_{Z/B}[1].$$

Applying q_* and taking weight-1 parts is an exact functor. Applied to the last two terms of (3.1) it yields

$$(3.2) \quad \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathbb{L}_{Z/B}[1]$$

on Z . We call this map $\mathrm{At}_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \mathbb{L}_{Z/B})$ the Atiyah class of \mathcal{F} .

We apply this to the universal sheaf \mathcal{E} on $\mathcal{N} \times_B X$ and project to the first summand of $\mathbb{L}_{\mathcal{N} \times_B X/B} = \mathbb{L}_{\mathcal{N}/B} \oplus \mathbb{L}_{X/B}$ (suppressing some pullback maps) to give the partial Atiyah class

$$(3.3) \quad \mathrm{At}_{\mathcal{E}, \mathcal{N}}: \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/B} \quad \text{on } \mathcal{N} \times_B X.$$

Similarly applied to E on $\mathcal{N} \times_B S$ and projecting $\mathbb{L}_{\mathcal{N} \times_B S}$ to $\mathbb{L}_{\mathcal{N}/B}$ defines a partial Atiyah class

$$(3.4) \quad \mathrm{At}_{E, \mathcal{N}}: E \longrightarrow E \otimes \mathbb{L}_{\mathcal{N}/B}[1] \quad \text{on } \mathcal{N} \times_B S.$$

The relationship between these two classes turns out to be very simple, and will allow us to relate the deformation theories of \mathcal{E} and E .

Proposition 3.5. *Applying π_* to (3.3),*

$$\pi_* \mathrm{At}_{\mathcal{E}, \mathcal{N}}: \pi_* \mathcal{E} \longrightarrow \pi_* \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/B}[1]$$

gives (3.4). That is, $\pi_ \mathrm{At}_{\mathcal{E}, \mathcal{N}} = \mathrm{At}_{E, \mathcal{N}}$.*

Proof. We use the natural \mathbb{C}^* -equivariant¹¹ map

$$(3.6) \quad \mathcal{N} \times_B X[\mathcal{E}] \xrightarrow{\rho} \mathcal{N} \times_B S[\mathbf{E}]$$

defined by thinking of both as affine schemes over $\mathcal{N} \times_B S$ and using the pullback map on (sheaves of $\mathcal{O}_{\mathcal{N} \times_B S}$ -) algebras:

$$(\mathcal{O}_S \oplus K_S^{-1} \oplus K_S^{-2} \oplus \cdots) \oplus \mathbf{E} \xleftarrow{(1,0,0,\dots,1)} \mathcal{O}_S \oplus \mathbf{E}.$$

(The algebra structure on the left hand side is defined by the action (2.7) of each K_S^{-i} on \mathbf{E} .) Combining the last two terms of (3.1) with pullback via ρ (3.6) gives the commutative diagram

$$\begin{array}{ccc} \mathbb{L}_{\mathcal{N} \times_B X[\mathcal{E}]/\mathcal{N} \times_B X} & \longrightarrow & q^* \mathbb{L}_{\mathcal{N} \times_B X}[1] \\ \rho^* \uparrow & & \uparrow \rho^* \\ \rho^* \mathbb{L}_{\mathcal{N} \times_B S[\mathbf{E}]/\mathcal{N} \times_B S} & \longrightarrow & \rho^* q^* \mathbb{L}_{\mathcal{N} \times_B S}[1]. \end{array}$$

Taking weight 1 parts gives

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\text{At}_{\mathcal{E}}} & \mathcal{E} \otimes \mathbb{L}_{\mathcal{N} \times_B X}[1] \\ \text{ev} \uparrow & & \text{ev} \otimes \uparrow \pi^* \\ \pi^* \mathbf{E} & \xrightarrow{\pi^* \text{At}_{\mathbf{E}}} & \pi^* \mathbf{E} \otimes \pi^* \mathbb{L}_{\mathcal{N} \times_B S}[1]. \end{array}$$

Projecting to the $\mathbb{L}_{\mathcal{N}/B}$ factor in both terms on the right hand side gives the commutative diagram

$$(3.7) \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\text{At}_{\mathcal{E}, \mathcal{N}}} & \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/B}[1] \\ \text{ev} \uparrow & & \text{ev} \otimes \uparrow 1 \\ \pi^* \mathbf{E} & \xrightarrow{\pi^* \text{At}_{\mathbf{E}, \mathcal{N}}} & \pi^* \mathbf{E} \otimes \mathbb{L}_{\mathcal{N}/B}[1]. \end{array}$$

Apply the adjunction isomorphism

$$(3.8) \quad \begin{array}{ccc} \text{Hom}(\pi^* \mathbf{E}, \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/B}[1]) & \cong & \text{Hom}(\mathbf{E}, \pi_* \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/B}[1]) \\ (\text{ev} \otimes 1) \circ \pi^* a & \longleftarrow & a \\ b \circ \text{ev} = (\text{ev} \otimes 1) \circ (\pi^* \pi_* b) & \longmapsto & \pi_* b \end{array}$$

to $a = \text{At}_{\mathbf{E}, \mathcal{N}}$ and $b = \text{At}_{\mathcal{E}, \mathcal{N}}$. The commutativity of (3.7) says that the two maps on the left hand side of (3.8) agree. Therefore the two maps on the right hand side agree. That is, $a = \pi_* b$ as required. \square

Consider that partial Atiyah class (3.3) as lying in the group

$$(3.9) \quad \begin{aligned} \text{At}_{\mathcal{E}, \mathcal{N}} &\in \text{Ext}_{\mathcal{N} \times_B X}^1(\mathcal{E}, \mathcal{E} \otimes p_X^* \mathbb{L}_{\mathcal{N}/B}) \\ &= \text{Ext}_{\mathcal{N} \times_B X}^1(R\mathcal{H}om(\mathcal{E}, \mathcal{E}), p_X^* \mathbb{L}_{\mathcal{N}/B}) \\ &\cong \text{Ext}_{\mathcal{N}}^2(Rp_{X*} R\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes K_{X/B}), \mathbb{L}_{\mathcal{N}/B}), \end{aligned}$$

¹¹Here we are using the \mathbb{C}^* action which acts with weight 1 on \mathcal{E} and \mathbf{E} but fixes X and S . Only later will we use the different \mathbb{C}^* action on the fibres of $\pi: X \rightarrow S$.

using relative Serre duality for the projection $p_X: \mathcal{N} \times_B X \rightarrow \mathcal{N}$, which has relative dimension 3. Of course the relative canonical bundle $K_{X/B} \cong \mathcal{O}$ is trivial, but not *equivariantly* with respect to the \mathbb{C}^* action scaling the fibres of $X \rightarrow S$. This acts on $K_{X/B}$ with weight -1 , so

$$K_{X/B} \cong \mathcal{O} \otimes \mathfrak{t}^{-1},$$

where \mathfrak{t} is the standard 1-dimensional representation of \mathbb{C}^* . All told we get a map

$$(3.10) \quad \text{At}_{\mathcal{E}, \mathcal{N}}: R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})\mathfrak{t}^{-1}[2] \longrightarrow \mathbb{L}_{\mathcal{N}/B}.$$

Theorem 3.11. *The map (3.10) is a relative obstruction theory for \mathcal{N}/B in the sense of [BF, Definition 4.4 and Section 7].*

Proof. Here we can ignore the \mathbb{C}^* action. Fix a morphism of B -schemes,

$$f: T_0 \longrightarrow \mathcal{N},$$

and an extension of B -schemes $T_0 \subset T$ with ideal I such that $I^2 = 0$.

Compose the pullback of (3.10),

$$(3.12) \quad f^* R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \longrightarrow f^* \mathbb{L}_{\mathcal{N}/B},$$

with the natural map $f^* \mathbb{L}_{\mathcal{N}/B} \rightarrow \mathbb{L}_{T_0/B}$ followed by the composition¹²

$$(3.13) \quad \mathbb{L}_{T_0/B} \longrightarrow \mathbb{L}_{T_0/T} \xrightarrow{\tau^{\geq -1}} \tau^{\geq -1} \mathbb{L}_{T_0/T} = I[1].$$

This defines an element

$$(3.14) \quad o \in \text{Ext}_{T_0}^{-1}(f^* R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}), I).$$

By [BF, Theorem 4.5] our Theorem will follow if we can show that o vanishes if and only if there exists an extension from T_0 to T of the map f of B -schemes, and that when $o = 0$ the set of extensions is a torsor under $\text{Ext}_{T_0}^{-2}(f^* R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}), I)$.

Let

$$\bar{f} = \text{id} \times f: X \times_B T_0 \longrightarrow X \times_B \mathcal{N}$$

and let \bar{p}_X be the projection

$$\bar{p}_X: X \times_B T_0 \longrightarrow T_0.$$

Since \bar{p}_X is flat,

$$(3.15) \quad f^* R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \cong R\mathcal{H}om_{\bar{p}_X}(\bar{f}^* \mathcal{E}, \bar{f}^* \mathcal{E}),$$

and, by the obvious functoriality of Atiyah classes, the pullback (3.12) of the partial Atiyah class is the partial Atiyah class $\text{At}_{\bar{f}^* \mathcal{E}, T_0}$. Therefore, unwinding the working above and the relative Serre duality of (3.9), we can write (3.14)

$$o \in \text{Ext}_{T_0}^{-1}(f^* R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}), I) \cong \text{Ext}_{X \times_B T_0}^2(\bar{f}^* \mathcal{E}, \bar{f}^* \mathcal{E} \otimes I)$$

¹²This composition $\mathbb{L}_{T_0/B} \rightarrow I[1]$ is the Kodaira-Spencer class of $T_0 \subset T$.

as the composition of

$$\mathrm{At}_{\bar{f}^*\mathcal{E}, T_0} \in \mathrm{Ext}^1(\bar{f}^*\mathcal{E}, \bar{f}^*\mathcal{E} \otimes \mathbb{L}_{T_0/B})$$

with the Kodaira-Spencer class (3.13). But in [Ill, Proposition III.3.1.8], Illusie shows that precisely this composition is his class $\omega(B, \mathcal{E})$, which in [Ill, Proposition III.3.1.5] he shows vanishes if and only if $\bar{f}^*\mathcal{E}$ deforms from $T_0 \times_B X$ to $T \times_B X$. Furthermore, when it does vanish, he shows such deformations form a torsor under $\mathrm{Ext}_{T_0 \times_B X}^1(\bar{f}^*\mathcal{E}, \bar{f}^*\mathcal{E} \otimes I)$, which by (3.15) and relative Serre duality for \bar{p}_X is $\mathrm{Ext}_{T_0}^{-2}(f^*R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}), I)$.

Since deformations of $\bar{f}^*\mathcal{E}$ from $X \times_B T_0$ to $X \times_B T$ are in 1-1 correspondence with extensions from T_0 to T of the B -map f , we are done. \square

Corollary 3.16. *There is a perfect relative obstruction theory of amplitude $[-1, 0]$ for \mathcal{N} ,*

$$(3.17) \quad \tau^{[-1, 0]}(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2])\mathfrak{t}^{-1} \longrightarrow \mathbb{L}_{\mathcal{N}/B}.$$

Proof. Again, for this proof we can ignore the \mathbb{C}^* action. The complex $R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})$ is perfect of amplitude $[0, 3]$. Since stable sheaves are automatically simple we have the isomorphism

$$(3.18) \quad \mathbb{C} \xrightarrow[\sim]{\mathrm{id}} \mathrm{Hom}_X(\mathcal{E}_t, \mathcal{E}_t)$$

for all closed points $t \in \mathcal{N}$. Therefore, by the standard Nakayama lemma arguments (see for instance [HT, Lemma 4.2]),

$$(3.19) \quad \mathrm{Cone}\left(\mathcal{O}_{\mathcal{N}} \xrightarrow{\mathrm{id}} R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})\right) = \tau^{\geq 1}R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})$$

is also perfect of amplitude $[1, 3]$.

Similarly, Serre dual to (3.18), we have the isomorphisms

$$\mathrm{Ext}^3(\mathcal{E}_t, \mathcal{E}_t) \xrightarrow[\sim]{\mathrm{tr}} \mathbb{C}$$

for all closed points $t \in \mathcal{N}$. (Serre duality holds despite the noncompactness of X because $\mathrm{supp} \mathcal{E}_t$ is proper. Alternatively, pushforward to the projective completion $\overline{X} = \mathbb{P}(\mathcal{O}_S \oplus K_S)$, which leaves the Ext groups unchanged. Then ordinary projective Serre duality on \overline{X} gives $\mathrm{Ext}^3(\mathcal{E}_t, \mathcal{E}_t) = \mathrm{Hom}(\mathcal{E}_t, \mathcal{E}_t \otimes K_{\overline{X}})^* = \mathrm{Hom}(\mathcal{E}_t, \mathcal{E}_t)^* = \mathbb{C}$ since the restriction of $K_{\overline{X}}$ to $X \supset \mathrm{supp} \mathcal{E}_t$ is trivial.) Therefore

$$\mathrm{Cone}\left(\tau^{\geq 1}R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathrm{tr}} \mathcal{O}_{\mathcal{N}}[-3]\right)[-1] = \tau^{[1, 2]}R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})$$

is perfect of amplitude $[1, 2]$. We will show that the perfect obstruction theory (3.10) factors through these two truncations.

By (the dual of) [STV, Proposition 3.2]¹³ we have a commutative diagram

$$\begin{array}{ccc} Rp_{X*}\mathcal{O}_{\mathcal{N} \times_B X}[2] & \xrightarrow{\text{id}} & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \\ \text{At}_{\det \mathcal{E}, \mathcal{N}} \downarrow & & \downarrow \text{At}_{\mathcal{E}, \mathcal{N}} \\ \mathbb{L}_{\text{Pic}(X/B)/B} & \xrightarrow{\det^*} & \mathbb{L}_{\mathcal{N}/B}, \end{array}$$

where $\det: \mathcal{N} \rightarrow \text{Pic}(X/B)$ takes \mathcal{E} to its determinant $\mathcal{O}_X(rS)$. Since this is the constant map,¹⁴ we conclude the composition

$$(3.20) \quad \mathcal{O}_{\mathcal{N}}[2] \xrightarrow{\text{id}} R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \xrightarrow{\text{At}_{\mathcal{E}, \mathcal{N}}} \mathbb{L}_{\mathcal{N}/B}$$

is zero. By (3.19) the perfect obstruction theory (3.10) therefore factors through a map

$$\tau^{\geq -1} \left(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \right) \longrightarrow \mathbb{L}_{\mathcal{N}/B}.$$

Composing with the natural map $\tau^{[-1,0]} \rightarrow \tau^{\geq -1}$ gives the required map

$$\tau^{[-1,0]} \left(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \right) \longrightarrow \mathbb{L}_{\mathcal{N}/B}.$$

Since we have only modified (3.10) in degrees -2 and $+1$, it still induces an isomorphism on cohomology in degree 0 and a surjection in degree -1 . Thus it is a perfect obstruction theory. \square

4. THE $U(r)$ VAFA-WITTEN INVARIANT

From now on we will forget about the base B , taking it to be a point for simplicity. It will reappear in Section 6, where it will prove deformation invariance of our Vafa-Witten invariants.

4.1. Localisation. The \mathbb{C}^* action of weight 1 on the fibres of $X = K_S \rightarrow S$, corresponding to the grading (2.5) on the sheaf of algebras $\pi_*\mathcal{O}$, induces a \mathbb{C}^* -action on \mathcal{N} , and the obstruction theory (3.17) is naturally \mathbb{C}^* -equivariant. By [GP] the \mathbb{C}^* -fixed locus $\mathcal{N}^{\mathbb{C}^*}$ inherits a perfect obstruction theory

$$(4.1) \quad \tau^{[-1,0]} \left(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] \mathfrak{t}^{-1} \right)^f \longrightarrow \mathbb{L}_{\mathcal{N}^{\mathbb{C}^*}}$$

by taking the fixed (weight 0) part $(\cdot)^f$ of (3.17). This defines a virtual cycle

$$[\mathcal{N}^{\mathbb{C}^*}]^{\text{vir}} \in H_*(\mathcal{N}^{\mathbb{C}^*}).$$

¹³By [STV, Remark A.1] their definition of Atiyah class coincides with Illusie's.

¹⁴We can even do without this fact; since $\text{Pic}(X/B) \rightarrow B$ is smooth, $\mathbb{L}_{\text{Pic}(X/B)/B}$ is concentrated only in degree 0 . Therefore the composition (3.20) is zero in degree -2 , as required.

The derived dual of the moving (nonzero weight) part of (3.17) is called the virtual normal bundle,

$$(4.2) \quad \begin{aligned} N^{\text{vir}} &:= (\tau^{[-1,0]}(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2]\mathbf{t}^{-1})^{\text{mov}})^\vee \\ &= \tau^{[0,1]}(R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[1])^{\text{mov}}. \end{aligned}$$

Here the second expression follows from \mathbb{C}^* equivariant Serre duality on X . If \mathcal{N} were compact, its perfect obstruction theory (3.17) of virtual dimension zero would define a zero dimension virtual cycle [BF]. Its length would be a deformation invariant integer which we could calculate by the localisation formula [GP] as

$$(4.3) \quad \int_{[\mathcal{N}^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})}.$$

Here we represent N^{vir} as a 2-term complex $\{V_0 \rightarrow V_1\}$ of locally free \mathbb{C}^* -equivariant sheaves *with nonzero weights* and define

$$e(N^{\text{vir}}) := \frac{c_{\text{top}}^{\mathbb{C}^*}(V_0)}{c_{\text{top}}^{\mathbb{C}^*}(V_1)} \in H^*(\mathcal{N}^{\mathbb{C}^*}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[t, t^{-1}],$$

where $t = c_1(\mathbf{t})$ is the generator of $H^*(B\mathbb{C}^*) = \mathbb{Z}[t]$ and $c_{\text{top}}^{\mathbb{C}^*}$ denotes the \mathbb{C}^* -equivariant top Chern class lying in the *localised* \mathbb{C}^* -equivariant cohomology

$$H_{\mathbb{C}^*}^*(\mathcal{N}^{\mathbb{C}^*}, \mathbb{Z}) \otimes_{\mathbb{Z}[t]} \mathbb{Q}[t, t^{-1}] \cong H^*(\mathcal{N}^{\mathbb{C}^*}) \otimes \mathbb{Q}[t, t^{-1}].$$

Since \mathcal{N} is noncompact but $\mathcal{N}^{\mathbb{C}^*}$ is compact, we instead take the residue integral (4.3) as a *definition*. In this noncompact setting it is only a *rational* number in general.

Preliminary definition 4.4. *Let S be a smooth projective complex surface with $h^{0,1}(S) = 0 = h^{0,2}(S)$. For (r, c_1, c_2) for which all Gieseker semistable Higgs sheaves are Gieseker stable we define*

$$\widetilde{\text{VW}}_{r, c_1, c_2}(S) := \int_{[\mathcal{N}^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \in \mathbb{Q}.$$

Note this is a *constant* Laurent polynomial in $\mathbb{Q}[t, t^{-1}]$ because \mathcal{N} has virtual dimension zero. Equivalently, over each component of $\mathcal{N}^{\mathbb{C}^*}$, the virtual dimension of the obstruction theory (4.1) equals $-\text{rank}(N^{\text{vir}})$. (Both can vary from component to component, but they jump together.) $\widetilde{\text{VW}}(S)$ is just a local DT invariant of X .

Remark 4.5. The reason this is only a sensible definition for surfaces with $h^{0,1}(S) = 0 = h^{0,2}(S)$ is that the obstruction sheaf of the obstruction theory (4.1) has a trivial $H^2(\mathcal{O}_S)$ summand, so the virtual cycle is zero whenever $h^{0,2}(S) > 0$. And when $h^{0,1}(S) > 0$ the invariance of the obstruction theory under tensoring by flat line bundles means the integrand is pulled back from a lower dimensional space $\mathcal{N}/\text{Jac}(S)$, so the integral vanishes.

For more general surfaces we would like to fix the determinant of E and make ϕ trace-free, replacing the groups $\text{Ext}_S^i(E, E)$, $\text{Ext}_S^i(E, E \otimes K_S)$ by their trace-free counterparts.¹⁵ In gauge theory language, we want to replace $U(r)$ Higgs bundles with $SU(r)$. So instead of working on X with the first term of (2.15, 2.19) we need to work on S with the other two terms of (2.15, 2.19). Thus we need to relate the deformations of \mathcal{E} to those of E and ϕ . Equivalently, we need to express the deformation theory of \mathcal{N} in terms of that of \mathcal{M} and the fibres of $\Pi: \mathcal{N} \rightarrow \mathcal{M}$ (2.18). This is what we do next.

5. DEFORMATION THEORY OF THE HIGGS FIELD

By Proposition 3.5, the diagram

$$\begin{array}{ccc} R\mathcal{H}om_{p_S}(\mathcal{E}, \mathcal{E})[1] & \xleftarrow{\pi_*} & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[1] \\ \text{At}_{\mathcal{E}, \mathcal{N}} \uparrow & & \uparrow \text{At}_{\mathcal{E}, \mathcal{N}} \\ \Pi^* \mathbb{T}_{\mathcal{M}} & \xleftarrow{\Pi_*} & \mathbb{T}_{\mathcal{N}} \end{array}$$

commutes. Dualising and taking cones gives, via the exact triangle (2.19),

$$(5.1) \quad \begin{array}{ccccc} R\mathcal{H}om_{p_S}(\mathcal{E}, \mathcal{E} \otimes K_S)[1] & \longrightarrow & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[2] & \longrightarrow & R\mathcal{H}om_{p_S}(\mathcal{E}, \mathcal{E})[2] \\ \text{At}_{\mathcal{E}, \mathcal{N}} \downarrow & & \downarrow \text{At}_{\mathcal{E}, \mathcal{N}} & & \downarrow \text{At}_{\mathcal{E}, \mathcal{N}/\mathcal{M}} \\ \Pi^* \mathbb{L}_{\mathcal{M}} & \xrightarrow{\Pi^*} & \mathbb{L}_{\mathcal{N}} & \longrightarrow & \mathbb{L}_{\mathcal{N}/\mathcal{M}}. \end{array}$$

The right hand vertical arrow — produced by filling in the cones — is the projection of $\text{At}_{\mathcal{E}, \mathcal{N}}$ from $\mathbb{L}_{\mathcal{N}}$ to $\mathbb{L}_{\mathcal{N}/\mathcal{M}}$, which factors through the top right hand term in the diagram since it is zero on the top left.

Now any fibre of $\Pi: \mathcal{N} \rightarrow \mathcal{M}$ — the space of (stable) Higgs fields ϕ for a fixed sheaf $E \in \mathcal{M}$ — can be considered to be a space of quotients $\pi^*E \rightarrow \mathcal{E} \rightarrow 0$ of π^*E . That is, \mathcal{N}/\mathcal{M} is part¹⁶ of a relative Quot scheme of π^*E . Since the kernel of the quotient $\pi^*E \rightarrow \mathcal{E}$ is $\pi^*E \otimes K_S^{-1}$ (2.12), its deformations and obstructions are governed by

$$(5.2) \quad \text{Hom}_X(\pi^*E \otimes K_S^{-1}, \mathcal{E}) \cong \text{Hom}_S(E, E \otimes K_S)$$

and

$$\text{Ext}^1(E, E \otimes K_S)$$

respectively. By Serre duality these are cohomologies of $(R\text{Hom}(E, E)[2])^\vee$, so this is compatible with the last vertical arrow of the above diagram. Next we check the arrow is what we expect, i.e. that the diagram induces the usual obstruction theory for the relative Quot scheme \mathcal{N}/\mathcal{M} .

¹⁵This is **not** the same as replacing $\text{Ext}_X^i(\mathcal{E}, \mathcal{E})$ by its trace-free version $\text{Ext}_X^i(\mathcal{E}, \mathcal{E})_0$. Instead one should use the kernel of the map $\text{Ext}_X^i(\mathcal{E}, \mathcal{E}) \rightarrow H^{1, i+1}(\overline{X})$ given by cup product with the Atiyah class on \overline{X} of the pushforward of \mathcal{E} followed by the trace map; see for instance [KT1, Equation 27]. It is simpler to interpret this on S .

¹⁶It is the Zariski open in Quot consisting of all quotients $\pi^*E \rightarrow \mathcal{E}$ which induce an isomorphism $E \rightarrow \pi_* \mathcal{E}$ on sections.

5.1. Deformations of quotients. Illusie describes a *reduced* Atiyah class for quotients (or, more generally, maps of modules) and relates it to his Atiyah class of (3.1, 3.2). In our situation we again consider $\mathcal{N} \times X[\mathcal{E}]$ with its projection to $\mathcal{N} \times X/\mathcal{M} \times X$. We now also consider $\mathcal{N} \times X[\pi^*\mathbf{E}]$ with the embedding

$$\mathcal{N} \times X[\mathcal{E}] \hookrightarrow \mathcal{N} \times X[\pi^*\mathbf{E}]$$

induced from the surjection $\pi^*\mathbf{E} \rightarrow \mathcal{E} \rightarrow 0$. It has ideal

$$(5.3) \quad \pi^*\mathbf{E} \otimes K_S^{-1},$$

the kernel (2.12) of $\pi^*\mathbf{E} \rightarrow \mathcal{E}$.

These maps of spaces induce a commutative diagram of exact triangles

$$(5.4) \quad \begin{array}{ccccc} \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}]/\mathcal{M} \times X} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}]/\mathcal{N} \times X} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X/\mathcal{M} \times X}[1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}]/\mathcal{M} \times X[\pi^*\mathbf{E}]} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}]/\mathcal{N} \times X[\pi^*\mathbf{E}]} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\pi^*\mathbf{E}]/\mathcal{M} \times X[\pi^*\mathbf{E}]}[1] \end{array}$$

on $X \times \mathcal{N}[\mathcal{E}]$. As in (3.1, 3.2), taking the degree 1 part of the pushdown to $\mathcal{N} \times X$ of the right hand square gives

$$(5.5) \quad \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/\mathcal{M}}[1] \\ \downarrow & & \parallel \\ \pi^*\mathbf{E} \otimes K_S^{-1}[1] & \longrightarrow & \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/\mathcal{M}}[1], \end{array}$$

where the bottom left hand term is the ideal (5.3). The lower horizontal arrow defines the reduced Atiyah class¹⁷ of the quotient $\pi^*\mathbf{E} \rightarrow \mathcal{E}$,

$$(5.6) \quad \begin{aligned} \mathrm{At}_{\Phi}^{\mathrm{red}} &\in \mathrm{Hom}\left(\pi^*\mathbf{E} \otimes K_S^{-1}, \mathcal{E} \otimes \mathbb{L}_{\mathcal{N}/\mathcal{M}}\right) \\ &\cong \mathrm{Hom}\left(\mathbf{E} \otimes K_S^{-1}, \mathbf{E} \otimes \mathbb{L}_{\mathcal{N}/\mathcal{M}}\right), \end{aligned}$$

where the isomorphism follows from $\pi_*\mathcal{E} \cong \mathbf{E}$.

Proposition 5.7. *The right hand arrow of (5.1) is $\mathrm{At}_{\Phi}^{\mathrm{red}}$.*

Proof. This is just the statement that (5.5) commutes, since, by construction, the upper horizontal arrow is the Atiyah class $\mathrm{At}_{\mathcal{E}, \mathcal{N}/\mathcal{M}}$ of \mathcal{E} . \square

That is, the obstruction theory on \mathcal{N}/\mathcal{M} induced from considering \mathcal{N} to be a moduli space of sheaves \mathcal{E} on X (i.e. the right hand arrow of (5.1)) is the same as the standard obstruction theory for quotients $\pi^*\mathbf{E} \rightarrow \mathcal{E} \rightarrow 0$ provided by the reduced Atiyah class.

¹⁷Also known as the second fundamental form of $\pi^*\mathbf{E} \otimes K_S^{-1} \hookrightarrow \pi^*\mathbf{E}$.

5.2. Deformations of Higgs fields. The alternative description of any fibre of $\mathcal{N} \rightarrow \mathcal{M}$ as a (linear!) space of Higgs fields $\phi \in \text{Hom}(E, E \otimes K_S)$ means it has tangent space

$$\text{Hom}(E, E \otimes K_S)$$

and similarly an obstruction space $\text{Ext}^1(E, E \otimes K_S)$. Of course these are the same groups as before (5.2), and putting them together in a family we will show (see (5.18) for instance) this description induces a perfect relative obstruction theory

$$(5.8) \quad R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)^\vee \longrightarrow \mathbb{L}_{\mathcal{N}/\mathcal{M}}.$$

If we can show the arrow in (5.8) is $\text{At}_\Phi^{\text{red}}$ (5.6) then by Proposition 5.7 this gives the same perfect obstruction theory as given by the right hand arrow of (5.1). This will be helpful since it is this description, as elements of a linear space, that is easier to work with in passing to trace-free Higgs fields, which are harder to describe from the Quot point of view.

5.3. Matrices. We start by showing the equivalence of the two obstruction theories in the case of Higgs fields over a point, i.e. matrices. (Everything else we need will follow from this case by working in families.) So we fix *vector spaces* E and L of rank r and 1 respectively, and let H denote the moduli space of Higgs fields ϕ on E . It is the vector space

$$H := \text{Hom}(E, E \otimes L).$$

By the spectral construction, each $\phi \in H$ is equivalent to a length- r torsion sheaf \mathcal{E}_ϕ on L with an exact sequence

$$0 \longrightarrow \pi^*(E \otimes L^{-1}) \longrightarrow \pi^*E \longrightarrow \mathcal{E}_\phi \longrightarrow 0.$$

Here π is the projection from L to a point. Letting η denote the tautological section of π^*L on L , the second arrow is $\pi^*\phi - \eta$.

At any point $\phi \in H$ consider the maps

$$(5.9) \quad H \cong \mathbb{T}_H|_\phi \xrightarrow{\text{At}_\phi^{\text{red}}} \text{Hom}(\pi^*(E \otimes L^{-1}), \mathcal{E}) \cong \text{Hom}(E, E \otimes L) = H,$$

where the the first isomorphism is the identification of a linear space with its tangent space. We want to show the composition (5.9) is the identity to deduce that the two descriptions of H — as a linear space, or as a space of quotients — give rise to the same description of the tangent space. (In fact we want the family version of this for the universal sheaf \mathcal{E}_Φ on $H \times L$, but since H is smooth its tangent complex \mathbb{T}_H is a vector bundle, and to show an endomorphism of a vector bundle on a smooth space is the identity it is sufficient to check it on restriction to any point.)

Lemma 5.10. *The composition (5.9) is the identity.*

Proof. Let R denote the dual numbers $\mathbb{C}[t]/(t^2)$. Fix any tangent vector $\dot{\phi} \in H \cong T_{\phi}H$, or equivalently a map $\text{Spec } R \rightarrow H$. This corresponds to the family $\phi + t\dot{\phi}$ of Higgs fields over $\text{Spec } R$, and so the family of quotients

$$(5.11) \quad 0 \longrightarrow \pi^*(E \otimes L^{-1}) \otimes R \xrightarrow{\pi^*(\phi+t\dot{\phi})-\eta} \pi^*E \otimes R \longrightarrow \mathcal{E}_t \longrightarrow 0$$

defined by the cokernel of the second arrow.

Illusie [Ill, Section IV.3.2] shows any such family (flat over R) defines a deformation class in

$$(5.12) \quad \text{Hom}(\pi^*(E \otimes L^{-1}), \mathcal{E} \otimes (t)) \cong \text{Hom}(\pi^*(E \otimes L^{-1}), \mathcal{E})$$

described as follows. We compare the constant quotient (the right hand horizontal arrow in the diagram below) with the varying quotient (5.11) (the left hand horizontal arrow):

$$\begin{array}{ccc}
 \pi^*(E \otimes L^{-1}) \cdot t & & \mathcal{E}_{\phi} \cdot t \\
 \downarrow & \searrow 0 & \downarrow \\
 \pi^*(E \otimes L^{-1}) \otimes R & \xrightarrow{\pi^*(\phi+t\dot{\phi})-\eta} & \pi^*E \otimes R \longrightarrow \mathcal{E}_{\phi} \otimes R \\
 \downarrow & \nearrow \text{dashed} & \downarrow \\
 \pi^*(E \otimes L^{-1}) & & \mathcal{E}_{\phi}
 \end{array}$$

Since $t^2 = 0$ the upper diagonal arrow is zero, so the horizontal composition induces the lower diagonal arrow. This has composition zero with the projection to \mathcal{E}_{ϕ} so lifts uniquely to a map to $\mathcal{E}_{\phi} \cdot t$. Splitting $\mathcal{E}_{\phi} \otimes R = \mathcal{E}_{\phi} \cdot t \oplus \mathcal{E}_{\phi}$ and $\pi^*E \otimes R = \pi^*E \cdot t \oplus \pi^*E$, we see from the diagram that this map is $\dot{\phi}$.

Since Illusie also shows the class is $\text{At}_{\phi}^{\text{red}} \cdot \dot{\phi}$, this proves the Lemma. \square

From this universal case, we can deduce the same result for any family of Higgs fields ϕ on the vector space E , parametrised by any (possibly singular) base \mathcal{H} . Let $f: \mathcal{H} \rightarrow H$ be its classifying map and let Φ denote the universal Higgs field.

Corollary 5.13. *The map*

$$\text{At}_{f*\Phi}^{\text{red}}: \mathbb{T}_{\mathcal{H}} \longrightarrow f^*(R\text{Hom}(E, E \otimes L) \otimes \mathcal{O}_{\mathcal{H}}) = H \otimes \mathcal{O}_{\mathcal{H}}$$

is the same as the derivative

$$Df: \mathbb{T}_{\mathcal{H}} \longrightarrow f^*\mathbb{T}_H = H \otimes \mathcal{O}_{\mathcal{H}}.$$

Proof. By Lemma 5.10 and the remark preceding it we have proved this for the universal family H itself (where Df is the identity). That is, $\text{At}_{\Phi}^{\text{red}} = \text{id}$.

The obvious functoriality $f^* \text{At}_{\Phi}^{\text{red}} = \text{At}_{f*\Phi}^{\text{red}}$ therefore gives $\text{At}_{f*\Phi}^{\text{red}} = f^* \text{id} = Df$, as required. \square

5.4. Higgs bundles. We can now deduce the same result in a *family* of vector spaces, i.e. when E is a vector bundle.

Suppose that \mathcal{M} parametrises only *vector bundles* on S (or shrink it so that it does). Let X be the total space of K_S as usual. From the universal bundle E on $\mathcal{M} \times S$ we form the vector bundle

$$H := \mathcal{H}om(E, E \otimes K_S) \xrightarrow{\rho} \mathcal{M} \times S.$$

Over $H \times_{\mathcal{M} \times S} \mathcal{M} \times X \xrightarrow{\pi} H$ we get a universal Higgs field Φ and, by the spectral construction, a universal quotient

$$0 \longrightarrow \pi^*(\rho^*E \otimes K_S^{-1}) \longrightarrow \pi^*(\rho^*E) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Using the smooth linear structure on the fibres of $H \rightarrow \mathcal{M} \times S$ gives the first isomorphism in the sequence of maps

$$(5.14) \quad \begin{aligned} \rho^*H &\cong \mathbb{T}_{H/\mathcal{M} \times S} \xrightarrow{\text{At}_{\Phi}^{\text{red}}} \pi_* \mathcal{H}om(\pi^*(\rho^*E \otimes K_S^{-1}), \mathcal{E}) \\ &\cong \mathcal{H}om(\rho^*E, \rho^*E \otimes K_S) \cong \rho^*H. \end{aligned}$$

Lemma 5.15. *The composition (5.14) is the identity.*

Proof. Lemma 5.10 shows that restricted to any closed point of H the composition is the identity. If \mathcal{M} is reduced this proves the claim.

In general it is sufficient to check the claim locally, since maps of sheaves (rather than complexes of sheaves) are equal if and only if they are equal locally. So, shrinking \mathcal{M} and S if necessary, we can assume that both E and K_S are trivial. Then applying Corollary 5.13 to f the projection $H \cong H \times \mathcal{M} \times S \rightarrow H$ proves that (5.14) is the identity. \square

Thinking of \mathcal{N}/\mathcal{M} as a moduli space of sections of $H \rightarrow \mathcal{M} \times S$, the graph of the universal section Φ gives an embedding

$$(5.16) \quad \mathcal{N} \times S \hookrightarrow \Pi^*H,$$

where $\Pi: \mathcal{N} \times S \rightarrow \mathcal{M} \times S$ is the projection. Its normal bundle is the fibrewise tangent bundle of $\Pi^*H \rightarrow \mathcal{N} \times S$, which by the linear structure is just Π^*H :

$$(5.17) \quad N_{\Phi} = \text{Cone} \left(\mathbb{T}_{\mathcal{N} \times S} \xrightarrow{D\Phi} \Phi^* \mathbb{T}_{\Pi^*H} \right) \cong \Phi^* \mathbb{T}_{\Pi^*H/\mathcal{N} \times S} \cong \Pi^*H.$$

Therefore, by [BF, Proposition 6.2] applied to $(\mathcal{M} \times S)$ -maps (i.e. sections) from $\mathcal{M} \times S$ to H we get the linear relative obstruction theory for \mathcal{N} promised in (5.8) by starting with the map

$$(5.18) \quad p_S^* \mathbb{T}_{\mathcal{N}/\mathcal{M}} \cong \mathbb{T}_{\mathcal{N} \times S/\mathcal{M} \times S} \xrightarrow{D\Phi} \Phi^* \mathbb{T}_{\Pi^*H/\mathcal{N} \times S} = N_{\Phi} \cong \Pi^*H$$

and applying adjunction:

$$(5.19) \quad \mathbb{T}_{\mathcal{N}/\mathcal{M}} \longrightarrow R p_{S*}(N_{\Phi}) \cong R p_{S*}(\Pi^*H) \cong R \mathcal{H}om_{p_S} (E, E \otimes K_S).$$

Proposition 5.20. *The relative perfect obstruction theory (5.19) is the same as the one given by the right hand arrow of (5.1).*

Proof. By Proposition 5.7 the right hand arrow of (5.1) is given by the reduced Atiyah class $\text{At}_\Phi^{\text{red}}$ of the universal quotient $\pi^*\mathbf{E} \rightarrow \mathcal{E}_\Phi$ on $\mathcal{N} \times X$. By construction, it is induced from the composition of the arrow (5.18) with the pullback by $\Phi^*\Pi^*$ of (5.14). By Lemma 5.15 the latter is the identity, so we are done. \square

5.5. Trace. Taking the trace of the section Φ (5.16) of \mathbf{H} we get a section $\text{tr } \Phi$ of K_S instead:

$$(5.21) \quad \mathcal{N} \times S \xrightarrow{\Phi} \Pi^*\mathbf{H} \xrightarrow{\text{tr}} \mathcal{N} \times K_S.$$

Replacing the above analysis (5.17, 5.18, 5.19) for the section (5.16) (the graph of Φ) by the (simpler) analysis for the section (5.21) (the graph of $\text{tr } \Phi$) gives a perfect relative obstruction theory

$$\mathbb{T}_{\mathcal{M} \times H^0(K_S)/\mathcal{M}} \longrightarrow R p_{S*} K_S$$

for the moduli space $\mathcal{M} \times \Gamma(K_S) \rightarrow \mathcal{M}$ of pairs (E, σ) , $\sigma \in \Gamma(K_S)$, together with a commutative diagram

$$(5.22) \quad \begin{array}{ccc} R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S) & \xrightarrow{\text{tr}} & R p_{S*} K_S \\ \text{At}_\Phi^{\text{red}} \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{N}/\mathcal{M}} & \xrightarrow{D(\text{tr } \Phi)} & (\text{tr } \Phi)^* \mathbb{T}_{\mathcal{M} \times \Gamma(K_S)/\mathcal{M}} \end{array}$$

compatible with the map

$$(5.23) \quad \begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathcal{M} \times \Gamma(K_S), \\ (E, \phi) & \longmapsto & (E, \text{tr } \phi). \end{array}$$

Moreover, identifying $\mathbb{T}_{\mathcal{M} \times \Gamma(K_S)/\mathcal{M}}$ with $\Gamma(K_S) \otimes \mathcal{O}$ using the obvious linear structure, the construction of the right hand arrow of (5.22) (i.e. the analogue of (5.17) for the section $\text{tr } \Phi$) shows that it is the canonical embedding

$$\Gamma(K_S) \otimes \mathcal{O} \xrightarrow{H^0} R\Gamma(K_S) \otimes \mathcal{O} \cong R p_{S*} K_S.$$

Taking co-cones in (5.22) gives the commutative diagram of exact triangles

$$(5.24) \quad \begin{array}{ccccc} R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)_0 & \longrightarrow & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S) & \xrightarrow{\text{tr}} & R p_{S*} K_S \\ \text{At}_0^{\text{red}} \uparrow & & \text{At}_\Phi^{\text{red}} \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{N}/\mathcal{M} \times \Gamma(K_S)} & \longrightarrow & \mathbb{T}_{\mathcal{N}/\mathcal{M}} & \xrightarrow{\text{tr } \Phi} & (\text{tr } \Phi)^* \mathbb{T}_{\mathcal{M} \times \Gamma(K_S)/\mathcal{M}} \end{array}$$

with At_0^{red} being trace-free component of $\text{At}_\Phi^{\text{red}}$ in the splitting of the top row.

Fitting this in with (5.1) we get the diagram of exact triangles

$$\begin{array}{ccccc}
 R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)_0 & \longleftrightarrow & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S) & \xleftrightarrow[\text{id}]{\text{tr}} & R\Gamma(K_S) \\
 \downarrow & & \downarrow & & \parallel \\
 (5.25) \quad R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})^0[1] & \longleftrightarrow & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})[1] & \longleftrightarrow & R\Gamma(K_S) \\
 \downarrow & & \downarrow & & \\
 R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E})[1] & \xlongequal{\quad} & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E})[1] & &
 \end{array}$$

(where $R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})^0$ is defined by taking cones in the central row) receiving maps from the diagram

$$\begin{array}{ccccc}
 \mathbb{T}_{\mathcal{N}/\mathcal{M} \times \Gamma(K_S)} & \longrightarrow & \mathbb{T}_{\mathcal{N}/\mathcal{M}} & \longrightarrow & \mathbb{T}_{\Gamma(K_S)} \\
 \downarrow & & \downarrow & & \parallel \\
 (5.26) \quad \mathbb{T}_{\mathcal{N}/\Gamma(K_S)} & \longrightarrow & \mathbb{T}_{\mathcal{N}} & \longrightarrow & \mathbb{T}_{\Gamma(K_S)} \\
 \downarrow & & \downarrow & & \\
 \mathbb{T}_{\mathcal{M}} & \xlongequal{\quad} & \mathbb{T}_{\mathcal{M}} & &
 \end{array}$$

with everything commutative.

5.6. Determinant. Having dealt with the trace of Higgs fields, we now also deal with the determinant of Higgs bundles on S . This is more standard. We start with the left hand column of (5.25), making it the central column of the following commutative diagram of exact triangles

$$\begin{array}{ccccc}
 R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)_0 & \xlongequal{\quad} & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E} \otimes K_S)_0 & & \\
 \downarrow & & \downarrow & & \\
 (5.27) \quad R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_{\perp}[1] & \longleftrightarrow & R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})^0[1] & \longleftrightarrow & R\Gamma(\mathcal{O}_S)[1] \\
 \downarrow & & \downarrow & & \parallel \\
 R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E})_0[1] & \longleftrightarrow & R\mathcal{H}om_{p_S}(\mathbf{E}, \mathbf{E})[1] & \longleftrightarrow & R\Gamma(\mathcal{O}_S)[1].
 \end{array}$$

Here the central row is defined in Corollary 2.25. We then get the left hand column by taking co-cones. We claim this diagram is mapped to commutatively by the diagram of exact triangles¹⁸

$$\begin{array}{ccccc}
 \mathbb{T}_{\mathcal{N}/\mathcal{M} \times \Gamma(K_S)} & \xlongequal{\quad} & \mathbb{T}_{\mathcal{N}/\mathcal{M} \times \Gamma(K_S)} & & \\
 \downarrow & & \downarrow & & \\
 \mathbb{T}_{\mathcal{N}/\Gamma(K_S) \times \text{Pic}(S)} & \longrightarrow & \mathbb{T}_{\mathcal{N}/\Gamma(K_S)} & \xrightarrow{\det_*} & \mathbb{T}_{\text{Pic}(S)} \\
 \downarrow & & \downarrow & & \parallel \\
 \mathbb{T}_{\mathcal{M}/\text{Pic}(S)} & \longrightarrow & \mathbb{T}_{\mathcal{M}} & \xrightarrow{\det_*} & \mathbb{T}_{\text{Pic}(S)}.
 \end{array}$$

¹⁸Here, as before, $\det: \mathcal{N} \rightarrow \text{Pic}(S)$ is the map which on points takes $(E, \phi) \mapsto \det E$.

The maps between central columns come from the Atiyah class maps between (5.26) and (5.25). Between the right hand columns we similarly take the Atiyah class of the line bundle $\det \mathcal{E}$. These give a commutative square by (the dual of) [STV, Proposition 3.2]. Taking co-cones gives the maps on the left hand column.

The splitting of the central row of (5.27) defines a splitting of the Atiyah class $\text{At}_{\mathcal{E}, \mathcal{N}}$ into components $(\text{At}_{\mathcal{E}, \mathcal{N}}^\perp, \text{At}_{\det \mathcal{E}, \mathcal{N}})$.

Proposition 5.28. *Over the open locus of Higgs bundles in \mathcal{N} , the map*

$$\text{At}_{\mathcal{E}, \mathcal{N}}^\perp: R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_\perp[2]\mathfrak{t}^{-1} \longrightarrow \mathbb{L}_{\mathcal{N}/\Gamma(K_S) \times \text{Pic}(S)}$$

defined above is a 2-term symmetric perfect relative obstruction theory for $\mathcal{N}/\Gamma(K_S) \times \text{Pic}(S)$.

Proof. By an easy diagram chase on long exact sequences of cohomology sheaves, if we have maps of triangles of this sort in which two of the maps define perfect (relative) obstruction theories, the third also defines a perfect (relative) obstruction theory.

That $R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_\perp[2]$ is 2-term follows from basechange just as in Corollary 3.16. Its symmetry was noted in Corollary 2.25. Recording the \mathbb{C}^* action explicitly as in Corollary 3.16 explains the \mathfrak{t}^{-1} . \square

5.7. Higgs sheaves. When the rank of the vector space jumps in the family — i.e. when E is a sheaf rather than a bundle — things are more complicated and we need to replace sheaves by locally-free resolutions.

We first shrink \mathcal{M} to the image of $\Pi: \mathcal{N} \rightarrow \mathcal{M}$. This ensures boundedness since \mathcal{N} is quasi-projective. As \mathcal{M} parameterises only torsion-free sheaves, these all have homological dimension ≤ 1 on S . Therefore the universal sheaf E also has homological dimension ≤ 1 .

Fix an ample line bundle $\mathcal{O}(1)$ on S (and use the same notation $\mathcal{O}(1)$ for its pullbacks to $\mathcal{M} \times S$ and $\mathcal{N} \times S$ as usual). Then for sufficiently large $N \gg 0$ we get a surjection

$$(5.29) \quad E_1 := (p_S^* p_{S*} E(N))(-N) \xrightarrow{\text{ev}} E \longrightarrow 0 \quad \text{on } \mathcal{M} \times S.$$

Since E is flat over \mathcal{M} , for $N \gg 0$ we deduce that this E_1 is locally free. Letting E_2 be the kernel of (5.29), it is also locally free since E has homological dimension ≤ 1 . So from now on we fix this two-term locally free resolution

$$(5.30) \quad 0 \longrightarrow E_2 \xrightarrow{d} E_1 \longrightarrow E \longrightarrow 0.$$

Its advantage is its functoriality, which implies that any (twisted, Higgs field) endomorphism of E induces compatible endomorphisms of the E_i . The universal case is over \mathcal{N}/\mathcal{M} , where the universal Higgs field Φ on (the

pullback to $\mathcal{N} \times S$ of) \mathbf{E} induces canonical maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\text{ev}) & \xrightarrow{d} & (p_S^* p_{S*} \mathbf{E}(N))(-N) & \xrightarrow{\text{ev}} & \mathbf{E} \longrightarrow 0 \\ & & \downarrow \Phi_2 & & \downarrow \Phi_1 := p_S^* p_{S*} \Phi & & \downarrow \Phi \\ 0 & \longrightarrow & \ker(\text{ev}) & \xrightarrow{d} & (p_S^* p_{S*} \mathbf{E}(N))(-N) & \xrightarrow{\text{ev}} & \mathbf{E} \longrightarrow 0, \end{array}$$

i.e. $\Phi_i \in \text{Hom}(\mathbf{E}_i, \mathbf{E}_i \otimes K_S)$ such that

$$(5.31) \quad d \circ \Phi_2 = \Phi_1 \circ d.$$

Moreover the Φ_i satisfying (5.31) and inducing Φ are *unique*: any other choices differ by an element of

$$\begin{aligned} \text{Hom}(\mathbf{E}_1, \mathbf{E}_2) &= \text{Hom}((p_S^* p_{S*} \mathbf{E}(N))(-N), \mathbf{E}_2) \\ &\cong \text{Hom}(p_{S*}(\mathbf{E}(N)), p_{S*}(\mathbf{E}_2(N))) = 0, \end{aligned}$$

since the choice of \mathbf{E}_1 ensures that $p_{S*}(\mathbf{E}_2(N)) = 0$. We can phrase this as saying that \mathcal{N} is cut out of

$$(5.32) \quad \mathcal{N} \subset \mathcal{N}_1 \times_{\mathcal{M}} \mathcal{N}_2,$$

by the equation (5.31). Here $\mathcal{N}_i \rightarrow \mathcal{M}$ denotes the moduli space of Higgs fields ϕ_i , with fibre over $E \in \mathcal{M}$ given by $\text{Hom}(E_i, E_i \otimes K_S)$ (where the E_i are the restriction to $S \times \{E\}$ of the \mathbf{E}_i (5.30)).

Using (5.30), $R\mathcal{H}om(\mathbf{E}, \mathbf{E})$ is computed by the total complex of

$$(5.33) \quad \begin{array}{ccc} \mathbf{E}_1^* \otimes \mathbf{E}_2 & \xrightarrow{d^* \otimes 1} & \mathbf{E}_2^* \otimes \mathbf{E}_2 \\ 1 \otimes d \downarrow & & \downarrow 1 \otimes d \\ \mathbf{E}_1^* \otimes \mathbf{E}_1 & \xrightarrow{d^* \otimes 1} & \mathbf{E}_2^* \otimes \mathbf{E}_1. \end{array}$$

Since the top left corner injects into $\mathbf{E}_1^* \otimes \mathbf{E}_1$ this is quasi-isomorphic to

$$\begin{aligned} & \frac{\mathbf{E}_1^* \otimes \mathbf{E}_1 \oplus \mathbf{E}_2^* \otimes \mathbf{E}_2}{(1 \otimes d, d^* \otimes 1) \mathbf{E}_1^* \otimes \mathbf{E}_2} \xrightarrow{(d^* \otimes 1, -1 \otimes d)} \mathbf{E}_2^* \otimes \mathbf{E}_1 \\ & \text{id} := (1, 1) \updownarrow \text{tr} := (\text{tr}, -\text{tr}) \\ & \mathcal{O}_{\mathcal{M} \times S} \end{aligned}$$

with the given trace and identity maps. The identity $\text{tr} \circ \text{id} = \text{rank} \neq 0$ gives the usual splitting

$$R\mathcal{H}om(E, E) \cong R\mathcal{H}om(E, E)_0 \oplus \mathcal{O}_{\mathcal{M} \times S}.$$

Over \mathcal{N} the universal Higgs field Φ defines a section $\text{tr} \Phi$:

$$\mathcal{N} \times S \xrightarrow{\Phi} \mathcal{H}om(\mathbf{E}, \mathbf{E} \otimes K_S) \xrightarrow{\mathcal{H}^0} R\mathcal{H}om(\mathbf{E}, \mathbf{E} \otimes K_S) \xrightarrow{\text{tr}} K_S,$$

where as usual we have abbreviated π^*E to E and also suppressed the pull-back on K_S . So we have all the same ingredients as we had in the Higgs bundles case, except we have not yet shown that the composition

$$(5.34) \quad \mathbb{T}_{\mathcal{N} \times S / \mathcal{M} \times S} \xrightarrow{\text{At}^{\text{red}}} R\mathcal{H}om(E, E \otimes K_S) \xrightarrow{\text{tr}} K_S$$

is the same as¹⁹

$$(5.35) \quad \mathbb{T}_{\mathcal{N} \times S / \mathcal{M} \times S} \xrightarrow{D(\text{tr } \Phi)} K_S.$$

To prove the equality of (5.34) and (5.35) we repeat our earlier analysis for Higgs fields on vector bundles to both (E_1, ϕ_1) and (E_2, ϕ_2) . Section 5.1 gives reduced Atiyah classes (5.6)

$$\text{At}_{\phi_i}^{\text{red}} \in R\text{Hom}\left(E_i \otimes K_S^{-1}, E_i \otimes \mathbb{L}_{\mathcal{N}_i / \mathcal{M}}\right), \quad i = 1, 2,$$

which we consider as maps

$$(5.36) \quad \text{At}_{\phi_i}^{\text{red}}: p_S^* \mathbb{T}_{\mathcal{N}_i / \mathcal{M}} \longrightarrow R\mathcal{H}om(E_i \otimes K_S^{-1}, E_i) \cong E_i^* \otimes E_i \otimes K_S.$$

They determine the reduced Atiyah class $\text{At}_{\Phi}^{\text{red}}$ of E as follows.

Proposition 5.37. *Using the resolution (5.30) to write $R\mathcal{H}om$ as a total complex (5.33), the reduced Atiyah class of E ,*

$$(5.38) \quad p_S^* \mathbb{T}_{\mathcal{N} / \mathcal{M}} \xrightarrow{\text{At}_{\Phi}^{\text{red}}} R\mathcal{H}om(E \otimes K_S^{-1}, E).$$

becomes

$$(5.39) \quad p_S^* \mathbb{T}_{\mathcal{N} / \mathcal{M}} \xrightarrow{\begin{pmatrix} 0 & \text{At}_{\phi_2}^{\text{red}} \\ \text{At}_{\phi_1}^{\text{red}} & 0 \end{pmatrix}} \begin{array}{ccc} E_1^* \otimes E_2 \otimes K_S & \xrightarrow{d^* \otimes 1} & E_2^* \otimes E_2 \otimes K_S \\ \downarrow 1 \otimes d & & \downarrow 1 \otimes d \\ E_1^* \otimes E_1 \otimes K_S & \xrightarrow{d^* \otimes 1} & E_2^* \otimes E_1 \otimes K_S, \end{array}$$

where $\text{At}_{\phi_i}^{\text{red}}$ denotes the reduced Atiyah class (5.36) pulled back to \mathcal{N} via (5.32).

Note that (5.39) really defines a map in the derived category to the total complex on the right hand side because $d \circ \text{At}_{\phi_2}^{\text{red}} = \text{At}_{\phi_1}^{\text{red}} \circ d$ by (5.31) and the functoriality of Atiyah classes under the pullback to $\mathcal{N} \subset \mathcal{N}_1 \times_{\mathcal{M}} \mathcal{N}_2$.

Proof. We follow the definition of reduced Atiyah class in (5.4). The maps of sheaves on $\mathcal{N} \times X$,

$$\begin{array}{ccccc} \pi^* E_2 & \longrightarrow & \pi^* E_1 & \longrightarrow & \pi^* E \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}, \end{array}$$

¹⁹In other words, we will not extend all of Proposition 5.20 from bundles to sheaves to give the full equivalence of the two natural perfect relative obstruction theories for $\mathcal{N} / \mathcal{M}$, as it is enough for our purposes to deal only with the trace.

induce maps of spaces

$$\begin{array}{ccccc} \mathcal{N} \times X[\pi^* \mathbf{E}_2] & \longleftarrow & \mathcal{N} \times X[\pi^* \mathbf{E}_1] & \longleftarrow & \mathcal{N} \times X[\pi^* \mathbf{E}] \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{N} \times X[\mathcal{E}_2] & \longleftarrow & \mathcal{N} \times X[\mathcal{E}_1] & \longleftarrow & \mathcal{N} \times X[\mathcal{E}], \end{array}$$

where the vertical maps are embeddings with ideals $\pi^* \mathbf{E}_2 \otimes K_S^{-1}$, $\pi^* \mathbf{E}_1 \otimes K_S^{-1}$ and $\pi^* \mathbf{E} \otimes K_S^{-1}$ respectively. These induce maps of cotangent complexes

$$\begin{array}{ccccc} \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}_2]/\mathcal{N} \times X[\pi^* \mathbf{E}_2]} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}_1]/\mathcal{N} \times X[\pi^* \mathbf{E}_1]} & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\mathcal{E}]/\mathcal{N} \times X[\pi^* \mathbf{E}]} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{\mathcal{N} \times X[\pi^* \mathbf{E}_2]/\mathcal{M} \times X[\pi^* \mathbf{E}_2]}[1] & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\pi^* \mathbf{E}_1]/\mathcal{M} \times X[\pi^* \mathbf{E}_1]}[1] & \longrightarrow & \mathbb{L}_{\mathcal{N} \times X[\pi^* \mathbf{E}]/\mathcal{M} \times X[\pi^* \mathbf{E}]}[1]. \end{array}$$

Taking weight 1 parts of their pushdowns to $\mathcal{N} \times X$ gives

$$\begin{array}{ccccc} \pi^* \mathbf{E}_2 \otimes K_S^{-1}[1] & \longrightarrow & \pi^* \mathbf{E}_1 \otimes K_S^{-1}[1] & \longrightarrow & \pi^* \mathbf{E} \otimes K_S^{-1}[1] \\ \downarrow \text{At}_{\phi_2}^{\text{red}} & & \downarrow \text{At}_{\phi_1}^{\text{red}} & & \downarrow \text{At}_{\Phi}^{\text{red}} \\ \mathbb{L}_{\mathcal{N}/\mathcal{M}} \otimes \mathcal{E}_2[1] & \longrightarrow & \mathbb{L}_{\mathcal{N}/\mathcal{M}} \otimes \mathcal{E}_1[1] & \longrightarrow & \mathbb{L}_{\mathcal{N}/\mathcal{M}} \otimes \mathcal{E}[1], \end{array}$$

which is equivalent, by adjunction and $\pi_* \mathcal{E}_i = E_i$, $\pi_* \mathcal{E} = E$, to the following diagram of horizontal exact triangles on $\mathcal{N} \times S$:

$$\begin{array}{ccccc} \mathbb{T}_{\mathcal{N}/\mathcal{M}} \otimes E_2 & \longrightarrow & \mathbb{T}_{\mathcal{N}/\mathcal{M}} \otimes E_1 & \longrightarrow & \mathbb{T}_{\mathcal{N}/\mathcal{M}} \otimes E \\ \downarrow \text{At}_{\phi_2}^{\text{red}} & & \downarrow \text{At}_{\phi_1}^{\text{red}} & & \downarrow \text{At}_{\Phi}^{\text{red}} \\ E_2 \otimes K_S & \longrightarrow & E_1 \otimes K_S & \longrightarrow & E \otimes K_S. \end{array}$$

The commutativity of this diagram gives the claimed result. \square

In particular

$$\begin{aligned} \text{tr} \circ \text{At}_{\Phi}^{\text{red}} &= \text{tr} \circ \text{At}_{\phi_1}^{\text{red}} - \text{tr} \circ \text{At}_{\phi_2}^{\text{red}} \\ &= D(\text{tr}(\phi_1)) - D(\text{tr}(\phi_2)) = D(\text{tr} \Phi) \end{aligned}$$

by applying (5.22, 5.23) to both Higgs bundles (E_i, ϕ_i) . Hence (5.34) equals (5.35) and Sections 5.5 and 5.6 go through for Higgs sheaves as well as bundles. Thus Proposition 5.28 holds for Higgs sheaves too.

6. THE $SU(r)$ VAFA-WITTEN INVARIANT

Now that we have Proposition 5.28 for all of \mathcal{N} , we can restrict to the fibre \mathcal{N}_L^\perp over

$$(L, 0) \in \text{Pic}(S) \times \Gamma(K_S)$$

to deduce the following.

Theorem 6.1. *The moduli space \mathcal{N}_L^\perp of stable Higgs sheaves (E, ϕ) with $\det E \cong L$ and trace-free $\phi \in \text{Hom}(E, E \otimes K_S)_0$ admits a 2-term symmetric perfect obstruction theory*

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_\perp[1]t^{-1} \longrightarrow \mathbb{L}_{\mathcal{N}_L^\perp}.$$

\square

Localising as in Section 4, we can now give a general definition of our Vafa-Witten invariant for any surface S (which agrees with Preliminary Definition 4.4 when $h^{0,1}(S) = 0 = h^{0,2}(S)$).

Definition 6.2. *Let S be a smooth projective complex surface, and fix (r, c_1, c_2) with $r > 0$ for which all Gieseker semistable Higgs sheaves are Gieseker stable. We define*

$$(6.3) \quad \text{VW}_{r,c_1,c_2}(S) := \int_{[(\mathcal{N}_{r,L,c_2}^\perp(S))^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \in \mathbb{Q},$$

where L is any line bundle on S with $c_1(L) = c_1$.

This is deformation invariant under deformations of S for which c_1 remains of type (1,1). More precisely, suppose $\mathcal{S} \rightarrow B$ is a smooth family of projective surfaces over a connected base B with a global class $c_1 \in H^{1,1}(\mathcal{S})$ whose restriction to any fibre \mathcal{S}_b , $b \in B$, we also denote by $c_1 \in H^{1,1}(\mathcal{S}_b)$.²⁰ Let \mathcal{L} be any line bundle over \mathcal{S} with $c_1(\mathcal{L}) = c_1$. Then just as in Section 3 we may do everything relative to B so that $\mathcal{N}_{\mathcal{L}}^\perp \rightarrow B$ has a perfect relative obstruction theory, inducing one on its \mathbb{C}^* -fixed locus by [GP]. Thus

$$\int_{[(\mathcal{N}_{\mathcal{L}_b}^\perp(\mathcal{S}_b))^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})}$$

is independent of $b \in B$ by [BF, Proposition 7.2] and conservation of number [Fu, Theorem 10.2].

The invariant need *not* be deformation invariant, however, under deformations of S for which the Hodge type of $c_1(L)$ does not remain of Hodge type (1,1). In this situation there are no sheaves with $\det E = L$ on the deformed surface, so the invariant becomes zero.

7. VANISHING THEOREM AND THE TWO FIXED LOCI

The integral (6.3) is over two types of fixed components, as mentioned in (1.9, 1.10).

7.1. The first fixed locus. The first has $\phi = 0$ and so gives the moduli space \mathcal{M}_L of Gieseker stable²¹ sheaves on S of fixed determinant L . Consider the dual of the top row of the diagram of Corollary 2.25. Its coboundary map is $[\cdot, \phi]$ (2.19) which vanishes when $\phi = 0$. Therefore the exact triangle splits the obstruction theory of Theorem 6.1

$$R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_\perp[2]\mathfrak{t}^{-1} \cong R\mathcal{H}om_{p_S}(E, E \otimes K_S)_0[1] \oplus R\mathcal{H}om_{p_S}(E, E)_0[2]\mathfrak{t}^{-1}$$

into fixed and moving parts. By [GP] the former defines a perfect obstruction theory for \mathcal{M}_L ,

$$(7.1) \quad E^\bullet := R\mathcal{H}om_{p_S}(E, E \otimes K_S)_0[1] \longrightarrow \mathbb{L}_{\mathcal{M}_L}.$$

²⁰Equivalently the classifying map from B to the moduli stack of surfaces S has image in the *Noether-Lefschetz locus* of (S, c_1) .

²¹Recall we are fixing (r, c_1, c_2) for which stability = semistability.

Here E^\bullet is quasi-isomorphic to a 2-term complex of \mathbb{C}^* -fixed locally free sheaves $E^{-1} \rightarrow E^0$. Similarly, dualising the moving parts gives the virtual normal bundle

$$N^{\text{vir}} = R\mathcal{H}om_{p_S}(E, E \otimes K_S)_0 \mathfrak{t} = E^\bullet \otimes \mathfrak{t}[-1].$$

Therefore the contribution of the fixed locus \mathcal{M}_L to the invariant (4.3) is

$$\begin{aligned} \int_{[\mathcal{M}_L]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} &= \int_{[\mathcal{M}_L]^{\text{vir}}} \frac{c_s^{\mathbb{C}^*}(E^0 \otimes \mathfrak{t})}{c_r^{\mathbb{C}^*}(E^{-1} \otimes \mathfrak{t})} \\ &= \int_{[\mathcal{M}_L]^{\text{vir}}} \frac{c_s(E^0) + tc_{s-1}(E^0) + \dots}{c_r(E^{-1}) + tc_{r-1}(E^{-1}) + \dots}, \end{aligned}$$

where r and s are the ranks of E^{-1} and E^0 respectively. The integrand is homogeneous of degree $s - r$ equal to the virtual dimension

$$(7.2) \quad \text{vd} = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S)$$

of the perfect obstruction theory (7.1) for \mathcal{M}_L . Therefore only the t^0 coefficient has the correct degree vd over \mathcal{M}_L to have nonzero integral against $[\mathcal{M}_L]^{\text{vir}}$, and we may set $t = 1$ in the above to give

$$e_2(\mathcal{M}_L) = \int_{[\mathcal{M}_L]^{\text{vir}}} \left[\frac{c_\bullet(E^0)}{c_\bullet(E^{-1})} \right]_{\text{vd}}.$$

Here $c_\bullet(\cdot)$ denotes the total Chern class. This is

$$(7.3) \quad \int_{[\mathcal{M}_L]^{\text{vir}}} c_{\text{vd}}(E^\bullet) \in \mathbb{Z},$$

the top (in a virtual, or derived sense) Chern class of the virtual cotangent bundle E^\bullet of \mathcal{M}_L . This is the Ciocan-Fontanine-Kapranov/Fantechi-Göttsche signed Euler characteristic of \mathcal{M}_L studied in [JT].

7.2. Vanishing theorem. Often the above integer is the *only* contribution.

Proposition 7.4. *If $\deg K_S \leq 0$ then any stable \mathbb{C}^* -fixed Higgs pair (E, ϕ) has Higgs field $\phi = 0$. The Vafa-Witten invariant (6.3) is then the signed virtual Euler characteristic (7.3) of the moduli space $\mathcal{M}_L(S)$ of stable sheaves on S with fixed determinant L .*

Proof. Since both $\ker \phi$ and $\text{im } \phi$ are ϕ -invariant subsheaves (of E and $E \otimes K_S$ respectively) Gieseker stability gives the following inequalities on reduced Hilbert polynomials,

$$(7.5) \quad p_E(n) < p_{\text{im } \phi}(n) < p_{E \otimes K_S}(n) \quad \forall n \gg 0,$$

unless $\text{im } \phi$ is 0 or all of $E \otimes K_S$. But $\deg K_S \leq 0$ implies that $p_{E \otimes K_S}(n) \leq p_E(n)$ for $n \gg 0$. So either $\phi = 0$ or ϕ is an isomorphism. Since ϕ is \mathbb{C}^* -fixed it has determinant 0, so it cannot be an isomorphism. \square

Remark 7.6. *If either $\deg K_S < 0$ or $K_S = \mathcal{O}_S$ then for any $E \in \mathcal{M}_L$*

$$\mathrm{Ext}^2(E, E)_0 \cong \mathrm{Hom}(E, E \otimes K_S)_0^* = 0.$$

In the first case this is because E is stable and $\deg E \otimes K_S < \deg E$ so the only map $E \rightarrow E \otimes K_S$ is zero. In the second case it is because stable sheaves E are simple. Therefore the obstruction space vanishes at any $E \in \mathcal{M}_L$, so \mathcal{M}_L is smooth of dimension vd (7.2). In particular, the signed virtual Euler characteristic (7.3) is just the signed topological Euler characteristic

$$\mathrm{VW}_L = \int_{\mathcal{M}_L} c_{\mathrm{vd}}(\Omega_{\mathcal{M}_L}) = (-1)^{\mathrm{vd}} e(\mathcal{M}_L).$$

The same proof with non-strict inequalities in (7.5) gives the following.

Proposition 7.7. *If $\deg K_S < 0$ then any semistable \mathbb{C}^* -fixed Higgs pair (E, ϕ) has Higgs field $\phi = 0$. \square*

7.3. The second fixed locus. Connected components of the second type (1.10), with $\phi \neq 0$, are more interesting. Since the \mathbb{C}^* -fixed stable sheaves \mathcal{E}_ϕ are simple, we can make them \mathbb{C}^* -equivariant by [Ko, Proposition 4.4].²² Thus E carries a \mathbb{C}^* -action $\psi: \mathbb{C}^* \rightarrow \mathrm{Aut}(E)$ such that

$$(7.8) \quad \psi_t \circ \phi \circ \psi_t^{-1} = t\phi.$$

Therefore $E = \bigoplus_i E_i$ splits into weight spaces E_i on which t acts as t^i . That is, with respect to this splitting we can write $\psi_t = \mathrm{diag}(t^i)$. By (7.8), this action acts on the Higgs field with weight 1. Conversely, a Higgs pair (E, ϕ) with a \mathbb{C}^* action on E whose induced action on ϕ has weight 1 clearly defines a fixed point of our original \mathbb{C}^* action. Thus ϕ acts blockwise through maps

$$(7.9) \quad \phi_i: E_i \longrightarrow E_{i-1}.$$

When the E_i have rank 1 they are twists of ideal sheaves by line bundles (since they are torsion-free by stability), and the maps (7.9) define nesting of these ideals. Therefore one can express the \mathbb{C}^* -fixed locus \mathcal{M}_2 in terms of nested Hilbert schemes on S . In particular one gets the virtual cycles on nested Hilbert schemes discovered recently in [GSY1, GSY2], at least when $h^{0,1}(S) = 0 = h^{0,2}(S)$. We explain an extended example in detail next, leaving the general case to [GSY1].

8. CALCULATIONS ON SURFACES WITH POSITIVE CANONICAL BUNDLE

Let $(S, \mathcal{O}_S(1))$ be a smooth, connected polarised surface with

- $b_1(S) = 0$, and
- a smooth nonempty connected canonical divisor $C \in |K_S|$, such that
- $L = \mathcal{O}_S$ is the only line bundle satisfying $0 \leq \deg L \leq \frac{1}{2} \deg K_S$,

where degree is defined by $\deg L = c_1(L) \cdot c_1(\mathcal{O}_S(1))$. Examples include

- the generic quintic surface in \mathbb{P}^3 ,

²²In [TT2, Proposition 5.1] we prove the same holds for general (not necessarily simple) \mathbb{C}^* -fixed Higgs pairs, which we apply in the semistable case.

- general type surfaces with $\text{Pic}(S) = \mathbb{Z} \cdot K_S$, and
- the blow up of K3 in a point.

More generally, we expect many general type surfaces with $b_1 = 0$, $p_g > 0$ to have enough deformations to move the Hodge structure on $H^2(S)$ so the only integral (1,1) classes are rational multiples of $c_1(S)$. Such surfaces therefore satisfy the above conditions if $c_1(S) \in H^2(S, \mathbb{Z})$ is primitive.

By adjunction the genus g of C is

$$g = 1 + c_1(S)^2$$

while the bundles K_S and K_S^2 have sections

$$h^0(K_S) = p_g(S) = \frac{1}{12}(c_1(S)^2 + c_2(S)) - 1$$

and

$$(8.1) \quad h^0(K_S^2) = P_2(S) = p_g(S) + g = \frac{1}{12}(13c_1(S)^2 + c_2(S)).$$

By the obvious long exact sequences and Serre duality it is easy to show they have *no higher cohomology*.

We consider \mathbb{C}^* -fixed rank 2 Higgs pairs (E, ϕ) of fixed determinant K_S in the second fixed locus \mathcal{M}_2 (1.10). As in Section 7.3 the \mathbb{C}^* action defines a splitting $E = \oplus_i E_i$ into summands E_i , which, by semistability, are torsion free and so have rank > 0 . Therefore they have rank 1, and there are only two of them:

$$E = E_i \oplus E_j,$$

with $i > j$ without loss of generality. Since the Higgs field has weight 1, it takes weight k to weight $k - 1$. It is also nonzero, so we must have $j = i - 1$ and the only nonzero component of ϕ maps E_i to E_{i-1} .

Let \mathfrak{t} denotes the one dimensional \mathbb{C}^* representation of weight 1. By tensoring E by \mathfrak{t}^{-i} (i.e. multiplying the \mathbb{C}^* action on E by $\lambda^{-i} \cdot \text{id}_E$) we may assume without loss of generality that $i = 0$ and $j = -1$. Considering ϕ as a weight 0 element of $\text{Hom}(E, E \otimes K_S) \otimes \mathfrak{t}$, we have

$$(8.2) \quad E = E_0 \oplus E_{-1} \quad \text{and} \quad \phi = \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix} \quad \text{for some} \quad \Phi: E_0 \longrightarrow E_{-1} \otimes K_S \cdot \mathfrak{t}.$$

In particular, $E_{-1} \subset E$ is a ϕ -invariant subsheaf, so by semistability

$$\deg E_{-1} \leq \deg E_0 = \deg K_S - \deg E_{-1}.$$

But the existence of the nonzero map $\Phi: E_0 \rightarrow E_{-1} \otimes K_S$ implies

$$\deg E_{-1} + \deg K_S \geq \deg E_0 = \deg K_S - \deg E_{-1}.$$

Together these give $0 \leq \deg E_{-1} \leq \frac{1}{2} \deg K_S$, which by our assumptions on S implies that $\det E_{-1} = \mathcal{O}_S$ and $\det E_0 = K_S$. Therefore

$$E_0 = I_0 \otimes K_S, \quad E_{-1} = I_1 \cdot \mathfrak{t}^{-1},$$

for some ideal sheaves I_i . Since Φ defines a nonzero map $I_0 \rightarrow I_1$ we must have $I_0 \subseteq I_1$. That is, denoting by Z_j the 0-dimensional subscheme defined by the ideal I_j ,

$$(8.3) \quad Z_1 \subseteq Z_0.$$

This can be done in families to prove the following scheme-theoretic description of the \mathbb{C}^* -fixed loci.

Lemma 8.4. *Fix $r = 2$, $c_1 = -c_1(S)$ and $c_2 \in \mathbb{Z}$. For $c_2 < 0$ the \mathbb{C}^* -fixed locus is empty. For $c_2 \geq 0$ it is the disjoint union of \mathcal{M}_{r,c_1,c_2} and the nested Hilbert schemes*

$$(8.5) \quad \mathcal{M}_2 \cong \bigsqcup_{i=0}^{\lfloor c_2(E)/2 \rfloor} S^{[i, c_2-i]}$$

of subschemes $Z_1 \subseteq Z_0 \subset S$ of total length

$$|Z_0| + |Z_1| = c_2(E). \quad \square$$

In general $S^{[i,j]}$ is singular, but connected. In particular we have the cases

- $c_2(E) < 0$. Then both \mathcal{M}_L and \mathcal{M}_2 are empty (the first by the Bogomolov inequality for stable sheaves). So the Vafa-Witten invariant vanishes.
- $c_2(E) = 0$. Then \mathcal{M}_2 is the single point with $E = K_S \oplus \mathcal{O} \cdot \mathfrak{t}^{-1}$ and $\phi: E \rightarrow E \otimes K_S \cdot \mathfrak{t}$ is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This Higgs pair already appears in [VW, Equation 2.70].
- $c_2(E) = 1$. Then $\mathcal{M} \cong S$ with $x \in S$ corresponding to $E = I_x \otimes K_S \oplus \mathcal{O} \cdot \mathfrak{t}^{-1}$ and Higgs field $\phi = \begin{pmatrix} 0 & 0 \\ \iota & 0 \end{pmatrix}$, where ι is the (twist by K_S of) the inclusion $I_x \hookrightarrow \mathcal{O}$.
- $c_2(E) = 2$. Now \mathcal{M}_2 has two components,

$$\mathcal{M}_2 \cong S^{[2]} \sqcup S.$$

The first $S^{[2]} := \text{Hilb}^2 S$ is similar to the previous example, with length-2 subschemes $Z \subset S$ replacing $\{x\} \subset S$. The second is a copy of S , with $x \in S$ corresponding to $E = I_x \otimes K_S \oplus I_x \cdot \mathfrak{t}^{-1}$ and Higgs field $\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

8.1. Horizontal terms. We will calculate first for empty Z_1 , i.e. $I_1 = \mathcal{O}$ and

$$E = I_0 \otimes K_S \oplus \mathcal{O} \cdot \mathfrak{t}^{-1}.$$

Even this case turns out to be both hard and interesting. It describes *all* \mathbb{C}^* -fixed points when $c_2 \leq 1$, by the Bogomolov inequality. For any $c_2 \geq 0$ it describes a nonempty connected component $S^{[n]} = S^{[0,n]}$ of the \mathbb{C}^* -fixed locus, as noted above. Here $n = c_2(E)$ denotes the length of Z_0 .

We saw in Section 2.3 that the deformation theory of (E, ϕ) is governed by the cone on

$$(8.6) \quad R\mathrm{Hom}(E, E)_0 \xrightarrow{[\cdot, \phi]} R\mathrm{Hom}(E, E \otimes K_S \otimes \mathfrak{t})_0.$$

Here we have been more explicit about the obvious \mathbb{C}^* action that acts in Section 2.3. With respect to this action, the Higgs field ϕ becomes of a weight 0.

The splitting of E induces a splitting of $\mathrm{Hom}(E, E)$ which we denote, in the obvious notation, by

$$\begin{pmatrix} \mathrm{Hom}(I_0 K_S, I_0 K_S) & \mathrm{Hom}(\mathcal{O} \cdot \mathfrak{t}^{-1}, I_0 K_S) \\ \mathrm{Hom}(I_0 K_S, \mathcal{O} \cdot \mathfrak{t}^{-1}) & \mathrm{Hom}(\mathcal{O} \cdot \mathfrak{t}^{-1}, \mathcal{O} \cdot \mathfrak{t}^{-1}) \end{pmatrix} = \begin{pmatrix} \mathbb{C} \cdot \mathrm{id}_{I_0} & H^0(I_0 K_S) \mathfrak{t} \\ 0 & \mathbb{C} \cdot \mathrm{id}_{\mathcal{O}} \end{pmatrix}.$$

Similarly, $\mathrm{Hom}(E, E \otimes K_S \cdot \mathfrak{t})$ splits as

$$\begin{pmatrix} \mathrm{Hom}(I_0 K_S, I_0 K_S^2 \cdot \mathfrak{t}) & \mathrm{Hom}(\mathcal{O} \cdot \mathfrak{t}^{-1}, I_0 K_S^2 \cdot \mathfrak{t}) \\ \mathrm{Hom}(I_0 K_S, K_S) & \mathrm{Hom}(\mathcal{O} \cdot \mathfrak{t}^{-1}, K_S) \end{pmatrix} = \begin{pmatrix} H^0(K_S) \mathfrak{t} & H^0(I_0 K_S^2) \mathfrak{t}^2 \\ \mathbb{C} \cdot \iota & H^0(K_S) \mathfrak{t} \end{pmatrix}.$$

Since $\phi = \begin{pmatrix} 0 & 0 \\ \iota & 0 \end{pmatrix}$, the map $[\cdot, \phi]$ between them acts by

$$\begin{pmatrix} a & s \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} s\iota & 0 \\ (b-a)\iota & -\iota s \end{pmatrix}.$$

Setting $b = -a$ gives the map on trace-free groups. We find the map $\mathrm{Hom}(E, E)_0 \rightarrow \mathrm{Hom}(E, E \otimes K_S \otimes \mathfrak{t})_0$ is injective and has cokernel

$$(8.7) \quad \frac{H^0(K_S)}{\iota H^0(I_0 K_S)} \cdot \mathfrak{t} \oplus H^0(I_0 K_S^2) \mathfrak{t}^2.$$

Next we compute $\mathrm{Ext}^1(E, E) = \mathrm{Ext}^1(E, E)_0$ as

$$\begin{pmatrix} \mathrm{Ext}^1(I_0 K_S, I_0 K_S) & \mathrm{Ext}^1(\mathcal{O} \cdot \mathfrak{t}^{-1}, I_0 K_S) \\ \mathrm{Ext}^1(I_0 K_S, \mathcal{O} \cdot \mathfrak{t}^{-1}) & \mathrm{Ext}^1(\mathcal{O} \cdot \mathfrak{t}^{-1}, \mathcal{O} \cdot \mathfrak{t}^{-1}) \end{pmatrix} = \begin{pmatrix} T_{Z_0} S^{[n]} & H^1(I_0 K_S) \mathfrak{t} \\ H^1(I_0 K_S^2)^* \mathfrak{t}^{-1} & 0 \end{pmatrix},$$

where in the bottom left entry we have used Serre duality on S .

Similarly $\mathrm{Ext}^1(E, E \otimes K_S \cdot \mathfrak{t}) = \mathrm{Ext}^1(E, E \otimes K_S)_0 \cdot \mathfrak{t}$ is

$$\begin{pmatrix} \mathrm{Ext}^1(I_0 K_S, I_0 K_S^2 \cdot \mathfrak{t}) & \mathrm{Ext}^1(\mathcal{O} \cdot \mathfrak{t}^{-1}, I_0 K_S^2 \cdot \mathfrak{t}) \\ \mathrm{Ext}^1(I_0 K_S, K_S) & \mathrm{Ext}^1(\mathcal{O} \cdot \mathfrak{t}^{-1}, K_S) \end{pmatrix} = \begin{pmatrix} (T_{Z_0}^* S^{[n]}) \mathfrak{t} & H^1(I_0 K_S^2) \mathfrak{t}^2 \\ H^1(I_0 K_S)^* & 0 \end{pmatrix}.$$

The map $[\cdot, \phi]$ between them acts by

$$\begin{pmatrix} v & s \\ f & 0 \end{pmatrix} \mapsto \begin{pmatrix} s\iota & 0 \\ -\iota v & -\iota s \end{pmatrix}.$$

Lemma 8.8. *This map vanishes.*

Proof. In the exact sequence

$$(8.9) \quad \mathrm{Hom}(I_0, \mathcal{O}_{Z_0}) \longrightarrow \mathrm{Ext}^1(I_0, I_0) \xrightarrow{\iota} \mathrm{Ext}^1(I_0, \mathcal{O})$$

the first arrow is an isomorphism: it is the identity map from $T_{Z_0} S^{[n]}$ to itself. Therefore the second arrow ι is zero, and $\iota v = 0$.

Similarly we can see $s\iota$ from

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{O}, I_0 K_S) & \xrightarrow{\iota^*} & \mathrm{Ext}^1(I_0, I_0 K_S), \\ s & \longmapsto & s\iota. \end{array}$$

But ι^* is the Serre dual of the second arrow in (8.9), which we saw was zero.

Finally ιs lies in $H^1(K_S)$, which vanishes since $b_1(S) = 0$. \square

So we are now ready to analyse the deformation-obstruction theory of (E, ϕ) . Let T^i be the cohomology groups of the cone (8.6). We have the exact sequence

$$\begin{aligned} 0 \rightarrow T^0 \rightarrow \mathrm{Hom}(E, E)_0 &\xrightarrow{[\cdot, \phi]} \mathrm{Hom}(E, E \otimes K_S)_0 \mathfrak{t} \rightarrow T^1 \rightarrow \mathrm{Ext}^1(E, E) \xrightarrow{[\cdot, \phi]} \\ &\mathrm{Ext}^1(E, E \otimes K_S) \mathfrak{t} \rightarrow T^2 \longrightarrow \mathrm{Ext}^2(E, E)_0 \xrightarrow{[\cdot, \phi]} \mathrm{Ext}^2(E, E \otimes K_S)_0 \mathfrak{t} \rightarrow T^3 \rightarrow 0. \end{aligned}$$

We have shown that the third arrow is injective, so $T^0 = 0$. This says that (E, ϕ) have no infinitesimal automorphisms, as we knew already from stability. By Serre duality, the third-from-last map is surjective and $T^3 = 0$. From (8.7) and Lemma 8.8 the sequence splits into

$$0 \rightarrow \frac{H^0(K_S)}{\iota H^0(I_0 K_S)} \cdot \mathfrak{t} \oplus H^0(I_0 K_S^2) \mathfrak{t}^2 \longrightarrow T^1 \longrightarrow \begin{pmatrix} T_{Z_0} S^{[n]} & H^1(I_0 K_S) \mathfrak{t} \\ H^1(I_0 K_S)^* \mathfrak{t}^{-1} & 0 \end{pmatrix} \rightarrow 0$$

and its Serre dual for T^2 :

$$(8.10) \quad 0 \rightarrow \begin{pmatrix} (T_{Z_0}^* S^{[n]}) \mathfrak{t} & H^1(I_0 K_S^2) \mathfrak{t}^2 \\ H^1(I_0 K_S)^* & 0 \end{pmatrix} \longrightarrow T^2 \longrightarrow \left(\frac{H^0(K_S)}{\iota H^0(I_0 K_S)} \right)^* \oplus H^0(I_0 K_S^2)^* \mathfrak{t}^{-1} \rightarrow 0.$$

8.2. The virtual cycle. In particular we see the fixed (weight 0) part of T^1 is just $T_{Z_0} S^{[n]}$, as expected.

The weight 1 part of T^1 comes from putting together $H^0(K_S)/\iota H^0(I_0 K_S)$ and $H^1(I_0 K_S)$. Together we claim these give $\Gamma(K_S|_{Z_0})$. Although we only need the K-theory class of T^1 to compute the invariant (6.3), for completeness we briefly describe how to see this claim.

Digression. We do this by working with the \mathbb{C}^* -invariant sheaf \mathcal{E}_ϕ on X which is equivalent, via the spectral construction, to the Higgs pair (E, ϕ) . It is

$$\mathcal{E}_\phi = I_{Z_0 \subset 2S} \otimes K_S,$$

the pushforward from $2S$ to X of the ideal sheaf of $Z_0 \subset 2S$, all twisted by K_S . (Here $2S \subset X$ is the thickening of the zero section $S \subset X$ defined by the ideal $I_{S \subset X}^2$.) Let s_0 denote the tautological section of $\pi^* K_S$ on $X = K_S$, vanishing on the zero section S , and fix any section s of K_S . The support of \mathcal{E}_ϕ is $2S = \{s_0^2 = 0\}$; a weight-1 deformation of this (parameterised by t) is given by

$$(8.11) \quad s_0^2 = t^2 s^2,$$

splitting $2S$ into the two sections $s_0 = \pm ts$. Taking the structure sheaf of this deformation and twisting by both K_S and the ideal sheaf

$$(8.12) \quad \pi^* I_{Z_0} + (s_0 - ts)$$

gives a corresponding deformation of \mathcal{E}_ϕ . However, away from Z_0 is is the *trivial* deformation to first order. That is, working over $\text{Spec } \mathbb{C}[t]/(t^2)$ we see that (8.11) is just $s_0^2 = 0$ — i.e. the same $2S$ as at $t = 0$. The deformation *does* move Z_0 to first order, however, if and only if $s \notin I_{Z_0}$, as can be seen from (8.12). This describes the $H^0(K_S)/H^0(I_0 K_S)$ part of the weight-1 part of the first order deformations T^1 .

Since the deformation is zero away from Z_0 we do not need s to be a global section of K_S . Any local section defined near Z_0 also defines the above deformation — glued to zero away from Z_0 . This describes *all* of the weight-1 part of the first order deformations T^1 as $\Gamma(K_S|_{Z_0})$ — both the deformations $H^0(K_S)/H^0(I_0 K_S)$ coming from global sections s , and the quotient $H^1(I_0 K_S)$ by these.

Now $\Gamma(K_S|_{Z_0})$ is the fibre over $Z_0 \in S^{[n]}$ of the rank n bundle $K_S^{[n]}$ on $S^{[n]}$, and this bundle is globally the weight 1 part of T^1 . Dually, the (fixed part of the) obstruction bundle on our smooth \mathbb{C}^* -fixed moduli space \mathcal{M}_2 is $(K_S^{[n]})^*$.

Therefore the virtual cycle that \mathcal{M}_2 inherits from [GP] is the Euler class of this obstruction bundle:

$$\begin{aligned} [\mathcal{M}_2]^{\text{vir}} &= e((K_S^{[n]})^*) \cap S^{[n]} \\ &= (-1)^n e(K_S^{[n]}) \cap S^{[n]}. \end{aligned}$$

Fix the section of K_S cutting out the smooth canonical curve C . It induces a section of $K_S^{[n]}$ with zero locus $C^{[n]} = \text{Sym}^n C \subset S^{[n]}$. Since this has the correct dimension, it is Poincaré dual to $e(K_S^{[n]})$, and we find that

$$(8.13) \quad [\mathcal{M}_2]^{\text{vir}} = (-1)^n [C^{[n]}] \subset S^{[n]} = \mathcal{M}_2.$$

8.3. The virtual normal bundle. Reading off the virtual normal bundle N^{vir} from the moving parts of the above computations (8.10) gives

$$\Gamma(K_S|_{Z_0})\mathfrak{t} \oplus R\Gamma(I_0 K_S^2)\mathfrak{t}^2 \oplus R\Gamma(I_0 K_S^2)^\vee \mathfrak{t}^{-1}[-1] \oplus T_{Z_0}^* S^{[n]}\mathfrak{t}[-1]$$

at $Z_0 \in \mathcal{M}_2$. Notice the final term has weight 1 — if Serre duality did not shift weights in this way, this term would have weight 0 and would be the fixed obstruction bundle. Therefore the virtual class $[\mathcal{M}_2]^{\text{vir}}$ would be a signed Euler characteristic as before. As it is, the \mathcal{M}_2 contribution is much more interesting.

We only care about its K-theory class, for which it simplifies things to express $R\Gamma(I_0 K_S^2)$ as $H^0(K_S^2) - H^0(K_S^2|_{Z_0})$. As Z_0 moves through $S^{[n]}$, the first term is a fixed \mathbb{C}^{P_2} , where the plurigenus $P_2(S) = h^0(K_S^2) = p_g(S) + g$

is the constant (8.1). The second term is the fibre of the vector bundle $(K_S^2)^{[n]}$. Therefore, in \mathbb{C}^* -equivariant K-theory, N^{vir} is

$$[K_S^{[n]}]t + (t^2)^{\oplus P_2} - [(K_S^2)^{[n]}]t^2 - (t^{-1})^{\oplus P_2} + [((K_S^2)^{[n]})^*]t^{-1} - [T_{S^{[n]}}^*]t.$$

Therefore

$$\begin{aligned} \frac{1}{e(N^{\text{vir}})} &= \frac{e((K_S^2)^{[n]}t^2)e((t^{-1})^{\oplus P_2})e(T_{S^{[n]}}^*t)}{e(K_S^{[n]}t)e((t^2)^{\oplus P_2})e(((K_S^2)^{[n]})^*t^{-1})} \\ &= \frac{(2t)^n c_{\frac{1}{2t}}((K_S^2)^{[n]}) \cdot (-t)^{P_2} \cdot t^{2n} c_{\frac{1}{t}}(T_{S^{[n]}}^*)}{t^n c_{\frac{1}{t}}(K_S^{[n]}) \cdot (2t)^{P_2} \cdot (-1)^n t^n c_{\frac{1}{t}}((K_S^2)^{[n]})} \\ (8.14) \quad &= (-2)^{n-P_2} t^n \frac{c_{\frac{1}{2t}}((K_S^2)^{[n]})c_{-\frac{1}{t}}(T_{S^{[n]}}^*)}{c_{\frac{1}{t}}(K_S^{[n]})c_{\frac{1}{t}}((K_S^2)^{[n]})}, \end{aligned}$$

where $c_s(E) := 1 + sc_1(E) + \dots + s^r c_r(E)$ for a bundle E of rank r ; when $s = 1$ this is c_\bullet (the total Chern class).

When we take the degree n part of this and integrate over the virtual cycle $(-1)^n [C^{[n]}]$ (8.13), only the t^0 part contributes because (8.14) has total degree n (that is $-\text{rank}(N^{\text{vir}}) = n$ equals the virtual dimension of \mathcal{M}_2). So we can set $t = 1$ to get the same answer. We can also use the fact that $C^{[n]}$ is cut out by a transverse section of $K_S^{[n]}$, so its normal bundle is the restriction of $K_S^{[n]}$. Thus $T_{S^{[n]}}|_{C^{[n]}} = T_{C^{[n]}} \oplus K_S^{[n]}|_{C^{[n]}}$ in K-theory, and (8.15)

$$\int_{[M_2]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} = (-2)^{-P_2} 2^n \int_{C^{[n]}} \frac{c_{\frac{1}{2}}((K_S^2)^{[n]})c_{-1}(T_{C^{[n]}})c_{-1}(K_S^{[n]})}{c_\bullet(K_S^{[n]})c_\bullet((K_S^2)^{[n]})}.$$

8.4. Tautological classes. To evaluate this integral we express it in terms of two tautological cohomology classes on $C^{[n]}$. Fix a basepoint $c_0 \in C$. The first class is

$$\omega := \text{PD}[C^{[n-1]}] \in H^2(C^{[n]}, \mathbb{Z}),$$

where $C^{[n-1]} \hookrightarrow C^{[n]}$ is the map taking $Z \mapsto Z + c_0$.

For the second we use the Abel-Jacobi map

$$(8.16) \quad \text{AJ}: C^{[n]} \longrightarrow \text{Pic}^n C, \quad Z \longmapsto \mathcal{O}(Z).$$

Tensoring with powers of $\mathcal{O}(c_0)$ makes the $\text{Pic}^n C$ isomorphic for all n , so we may pull back the theta divisor from $\text{Pic}^{g-1} C$. Its cohomology class

$$\theta \in H^2(\text{Pic}^n C, \mathbb{Z}) \cong \text{Hom}(\Lambda^2 H^1(C, \mathbb{Z}), \mathbb{Z})$$

maps $P_2, \beta \in H^1(C, \mathbb{Z})$ to $\int_C P_2 \wedge \beta$. We also use θ to denote its pullback $\text{AJ}^* \theta$, giving our second tautological class

$$(8.17) \quad \theta \in H^2(C^{[n]}, \mathbb{Z}).$$

The integrals of these classes are given by

$$\begin{aligned}
 \int_{C^{[n]}} \frac{\theta^i}{i!} \omega^{n-i} &= \int_{C^{[i]}} \frac{\theta^i}{i!} = \int_{\text{Pic}^i C} \text{AJ}_* [C^{[i]}] \cup \frac{\theta^i}{i!} = \int_{\text{Pic}^i C} \frac{\theta^{g-i}}{(g-i)!} \cdot \frac{\theta^i}{i!} \\
 (8.18) \quad &= \binom{g}{i}.
 \end{aligned}$$

The first equality follows from repeated intersection with the divisor $C^{[n-1]} \subset C^{[n]}$; the last two are Poincaré's formula [ACGH, Section I.5].

In our application we also need to know the two identities

$$c_t(T_{C^{[n]}}) = (1 + \omega t)^{n+1-g} \exp\left(\frac{-t\theta}{1 + \omega t}\right)$$

and

$$c_t(L^{[n]}) = (1 - \omega t)^{n+g-1-\deg L} \exp\left(\frac{t\theta}{1 - \omega t}\right)$$

from, for example, [ACGH, Section VIII.2].

Plugging these into (8.15) makes the integrand $(-2)^{-P_2} 2^n$ times by

$$\frac{(1 - \frac{\omega}{2})^{n+1-g} \exp\left(\frac{\theta}{2-\omega}\right) (1 - \omega)^{n+1-g} \exp\left(\frac{\theta}{1-\omega}\right) (1 + \omega)^n \exp\left(\frac{-\theta}{1+\omega}\right)}{(1 - \omega)^n \exp\left(\frac{\theta}{1-\omega}\right) (1 - \omega)^{n+1-g} \exp\left(\frac{\theta}{1-\omega}\right)}.$$

Therefore our invariant is

$$\begin{aligned}
 (8.19) \quad &(-2)^{-p_g(S)-1} (-1)^n \int_{C^{[n]}} (\omega - 2)^{n+1-g} \frac{(1 + \omega)^n}{(1 - \omega)^n} \exp\left(\frac{\theta}{2 - \omega} - \frac{\theta}{1 + \omega} - \frac{\theta}{1 - \omega}\right).
 \end{aligned}$$

To simplify this integral we notice from (8.18) that

$$\int_{C^{[n]}} \frac{\theta^i}{i!} \omega^{n-i} = \int_{C^{[n]}} \binom{g}{i} \omega^i \omega^{n-i},$$

so whenever $\theta^i/i!$ is integrated against only powers of ω we may replace it by $\binom{g}{i} \omega^i$. In particular,

$$\exp(\alpha\theta) = \sum_{i=0}^{\infty} \alpha^i \frac{\theta^i}{i!} \sim \sum_{i=0}^{\infty} \alpha^i \binom{g}{i} \omega^i = (1 + \alpha\omega)^g$$

for α a power series in ω . The \sim becomes an equality if we integrate both sides over $C^{[n]}$ against a power series in ω . In (8.19) this gives

$$\begin{aligned}
 &\frac{(-1)^{n+p_g(S)+1}}{2^{p_g(S)+1}} \int_{C^{[n]}} (\omega - 2)^{n+1-g} \frac{(1 + \omega)^n}{(1 - \omega)^n} \left(1 + \frac{\omega}{2 - \omega} - \frac{\omega}{1 + \omega} - \frac{\omega}{1 - \omega}\right)^g \\
 &= (-2)^{-p_g(S)-1} (-1)^n \int_{C^{[n]}} (\omega - 2)^{n+1-2g} \frac{(1 + \omega)^{n-g}}{(1 - \omega)^{n+g}} (4\omega - 2)^g.
 \end{aligned}$$

Therefore, in terms of generating series,

$$\begin{aligned}
 (-2)^{p_g(S)-g+1}(-1)^n \sum_{n=0}^{\infty} q^n \int_{C^{[n]}} \frac{1}{e(N^{\text{vir}})} \\
 (8.20) \quad \quad \quad = \sum_{n=0}^{\infty} q^n \int_{C^{[n]}} (\omega - 2)^{n+1-2g} \frac{(1+\omega)^{n-g}}{(1-\omega)^{n+g}} (1-2\omega)^g.
 \end{aligned}$$

8.5. The answer in closed form. The series (8.20) is the *diagonal* [St, Section 6.3] of the double generating series

$$(8.21) \quad \sum_{i,n=0}^{\infty} x^i t^n \int_{C^{[i]}} (\omega - 2)^{n+1-2g} \frac{(1+\omega)^{n-g}}{(1-\omega)^{n+g}} (1-2\omega)^g.$$

That is, if we write (8.21) as $\sum_{i,n} a_{in} x^i t^n$, then our generating series (8.20) is $\sum_n a_{nn} q^n$.

We can evaluate (8.21) by first summing over n to give

$$\sum_{i=0}^{\infty} x^i \int_{C^{[i]}} \frac{(1-2\omega)^g}{(\omega - 2)^{2g-1}(1-\omega^2)^g} \left(1 - t \frac{(\omega - 2)(1 + \omega)}{1 - \omega} \right)^{-1}.$$

Since the integrand is independent of i , and $\int_{C^{[i]}} \omega^j = \delta_{ij}$, the operator $\sum_{i=0}^{\infty} x^i \int_{C^{[i]}}$ simply acts by setting ω to be x . So we get

$$\frac{(1-2x)^g}{(x-2)^{2g-1}(1-x^2)^g} \frac{1-x}{1-x-t(x^2-x-2)}.$$

To find the diagonal (8.20) of this series we substitute $t = q/x$ and consider the integral [St, Section 6.3]

$$\frac{1}{2\pi i} \oint \frac{(1-2x)^g}{(x-2)^{2g-1}(1-x^2)^g} \frac{1-x}{1-x-\frac{q}{x}(x^2-x-2)} \frac{dx}{x}$$

around a small loop containing only those poles *which tend to the origin as* $q \rightarrow 0$. Thus (8.20) is the residue of

$$\frac{(1-2x)^g}{(x-2)^{2g-1}(1-x^2)^g} \frac{-(1-x)}{(1+q)x^2 - (1+q)x - 2q}$$

at the root

$$(8.22) \quad x_0 := \frac{1}{2} \left(1 - \sqrt{1 + \frac{8q}{1+q}} \right)$$

of the quadratic $(1+q)x^2 - (1+q)x - 2q$ in x . This is

$$\frac{(1-2x_0)^g}{(x_0-2)^{2g-1}(1+x_0)^g(1-x_0)^{g-1}} \frac{-1}{(1+q)(x_0-x_1)},$$

where $x_1 = \frac{1}{2} \left(1 + \sqrt{1 + \frac{8q}{1+q}} \right)$ is the other root. Substituting in $x_1 - x_0 = \sqrt{1 + \frac{8q}{1+q}}$ and $(1 - 2x_0)^g = \left(1 + \frac{8q}{1+q} \right)^{g/2}$, and multiplying out the denominator this becomes

$$\frac{(-1)^{g-1}}{(x_0^2 - x_0 - 2)^g (x_0^2 - 3x_0 + 2)^{g-1}} \frac{(1 + 9q)^{\frac{g-1}{2}}}{(1 + q)^{\frac{g+1}{2}}}.$$

Since x_0 was chosen as a root of $x_0^2 - x_0 - \frac{2q}{1+q} = 0$, and has the formula (8.22), we find the terms in the denominator are

$$x_0^2 - x_0 - 2 = -\frac{2}{1+q} \quad \text{and} \quad x_0^2 - 3x_0 + 2 = \frac{1 + 3q + \sqrt{(1+q)(1+9q)}}{1+q}.$$

Rearranging and substituting into (8.20) finally gives the answer.

Proposition 8.23.

$$\sum_{n=0}^{\infty} q^n \int_{C^{[n]}} \frac{1}{e(N^{\text{vir}})} = c(1-q)^{g-1} \left(1 + \frac{1-3q}{\sqrt{(1-q)(1-9q)}} \right)^{1-g},$$

where $c = (-1)^{p_g(S)+g} \cdot 2^{-p_g(S)-1}$. Expanding in powers of q gives

$$(8.24) \quad (-2)^{-p_g(S)-g} \left(1 - 2(g-1)q + (g-1)(2g-11)q^2 - \frac{2}{3}(g-1)(2g^2 - 31g + 126)q^3 + \dots \right).$$

Even after multiplying by a shift q^{-s} , this is clearly not modular [BE].

8.6. Vertical terms. Having dealt with the $\text{length}(Z_1) = 0$ component of \mathcal{M}_2 , we now turn to the other extreme: the $\text{length}(Z_1) = \text{length}(Z_0)$ component. In the description of Lemma 8.4, the Higgs field $\Phi: I_0 \rightarrow I_1$ must be an isomorphism, so we get the Hilbert scheme

$$S^{[n,n]} \cong S^{[n]}$$

of Higgs pairs

$$E = I_Z \otimes K_S \oplus I_Z \cdot \mathfrak{t}^{-1}, \quad \phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : E \rightarrow E \otimes K_S \cdot \mathfrak{t},$$

where $Z \subset S$ is a 0-dimensional subscheme of length $n = c_2(E)/2$. The torsion sheaf \mathcal{E}_ϕ on X that this corresponds to is the twist by $\pi^* K_S$ of

$$(8.25) \quad \mathcal{F}_Z := (\pi^* I_Z) \otimes \mathcal{O}_{2S}.$$

Here as usual $\pi: X \rightarrow S$ is the projection, and $2S \subset X$ is the doubling of the zero section.

To work out the deformation theory it is convenient to work with the description (8.25). By the exact sequence

$$0 \longrightarrow \pi^* I_Z(-2S) \longrightarrow \pi^* I_Z \longrightarrow \mathcal{F}_Z \longrightarrow 0$$

we find

$$R\mathrm{Hom}(\mathcal{F}_Z, \mathcal{F}_Z) \longrightarrow R\mathrm{Hom}(\pi^* I_Z, \mathcal{F}_Z) \longrightarrow R\mathrm{Hom}(pi^* I_Z, \pi_* \mathcal{F}_Z(2S)).$$

The section of $\mathcal{O}(2S)$ cutting out $2S \subset X$ annihilates \mathcal{F}_Z , so the second arrow is zero. By adjunction and $\pi_* \mathcal{F} = I_Z \oplus I_Z \otimes K_S^{-1} \cdot \mathfrak{t}^{-1}$ we therefore find

$$\begin{aligned} R\mathrm{Hom}_X(\mathcal{F}_Z, \mathcal{F}_Z) &\cong R\mathrm{Hom}_S(I_Z, I_Z) \oplus R\mathrm{Hom}_S(I_Z, I_Z \otimes K_S^{-1}) \mathfrak{t}^{-1} \\ &\oplus R\mathrm{Hom}_S(I_Z, I_Z \otimes K_S^2) \mathfrak{t}^2[-1] \oplus R\mathrm{Hom}_S(I_Z, I_Z \otimes K_S) \mathfrak{t}[-1]. \end{aligned}$$

The $SU(2)$ perfect obstruction theory $R\mathrm{Hom}_X(\mathcal{F}_Z, \mathcal{F}_Z)_\perp[1]$ (the derived dual of Theorem 6.1) comes from taking trace-free parts of the first and last terms, so $\mathrm{Hom}_\perp = 0 = \mathrm{Ext}_\perp^3$,

$$\begin{aligned} \mathrm{Ext}_X^1(\mathcal{F}_Z, \mathcal{F}_Z)_\perp &= \mathrm{Ext}_S^1(I_Z, I_Z) \oplus \mathrm{Ext}_S^1(I_Z, I_Z \otimes K_S^{-1}) \mathfrak{t}^{-1} \\ (8.26) \quad &\oplus \mathrm{Hom}_S(I_Z, I_Z \otimes K_S^2) \mathfrak{t}^2 \oplus \mathrm{Hom}_S(I_Z, I_Z \otimes K_S) \mathfrak{t}, \end{aligned}$$

and Ext_\perp^2 is its dual tensored with \mathfrak{t}^{-1} (by Serre duality and $\omega_X \cong \mathcal{O}_X \cdot \mathfrak{t}^{-1}$). The first term of (8.26) is just $T_Z S^{[n]}$, the fixed part of the deformations. The last term vanishes, so by duality so does the fixed part of the obstructions. Therefore

$$[S^{[n,n]}]^{\mathrm{vir}} = [S^{[n]}]$$

and we can read off the virtual normal bundle:

$$\begin{aligned} N^{\mathrm{vir}} &= \mathrm{Ext}_S^1(I_Z, I_Z \otimes K_S^{-1}) \mathfrak{t}^{-1} \oplus \mathrm{Hom}_S(I_Z, I_Z \otimes K_S^2) \mathfrak{t}^2 \\ &- \left[\mathrm{Ext}_S^1(I_Z, I_Z \otimes K_S) \mathfrak{t} \oplus \mathrm{Ext}_S^1(I_Z, I_Z \otimes K_S^2) \mathfrak{t}^2 \oplus \mathrm{Ext}_S^2(I_Z, I_Z \otimes K_S^{-1}) \mathfrak{t}^{-1} \right]. \end{aligned}$$

Integrating the reciprocal of its equivariant Euler class over $S^{[n]}$ is a project for future work, but in the case $n = 1$ it is easy enough. We get

$$\int_S \frac{1}{e(N^{\mathrm{vir}})} = \int_S \frac{e(\Omega_S \mathfrak{t}) e(T_S \otimes K_S^2 \mathfrak{t}^2) e(H^0(K_S^2)^* \mathfrak{t}^{-1})}{e(T_S \otimes K_S^{-1} \mathfrak{t}^{-1}) e(H^0(K_S^2) \mathfrak{t}^2)}.$$

Here we have repeatedly used the isomorphism $\mathrm{Ext}^1(I_p, I_p) \cong T_p S$ and its Serre dual, for any point $p \in S$. We have also computed $\mathrm{Ext}^1(I_p, I_p \otimes K_S^2) = T_p S \otimes K_S^2$ from the local-to-global spectral sequence $\oplus_{i+j=k} H^i(\mathcal{E} x t^j) \implies \mathrm{Ext}^k$ (using that $H^1(\mathcal{H}om) = H^2(\mathcal{H}om) = 0$ in this case). It follows from Serre duality that $\mathrm{Ext}^1(I_p, I_p \otimes K_S^{-1}) = T_p S \otimes K_S^{-1}$ (proving directly the vanishing of the differential $H^0(\mathcal{E} x t^1) \rightarrow H^2(\mathcal{H}om)$ would be more troublesome in this case).

In the notation of (8.14) this is

$$\int_S \frac{t^2 c_{\frac{1}{t}}(\Omega_S) \cdot (2t)^2 c_{\frac{1}{2t}}(T_S \otimes K_S^2) \cdot (-t)_2^P}{(-t)^2 c_{-\frac{1}{t}}(T_S \otimes K_S^{-1}) \cdot (2t)_2^P}.$$

Since only the t^0 term contributes, we may set $t = 1$ to give

$$(-2)^{-P_2} \int_S \frac{(1 - c_1 + c_2) \cdot 4 \left(1 + \frac{1}{2} c_1 (T_S \otimes K_S^2) + \frac{1}{4} c_2 (T_S \otimes K_S^2) \right)}{1 - c_1 (T_S \otimes K_S^{-1}) + c_2 (T_S \otimes K_S^{-1})},$$

where $c_i := c_i(S)$. Substituting in the identities

$$\begin{aligned} c_1(T_S \otimes K_S^2) &= c_1 + 2(-2c_1) = -3c_1, \\ c_2(T_S \otimes K_S^2) &= c_2 + c_1(-2c_1) + (-2c_1)^2 = c_2 + 2c_1^2, \\ c_1(T_S \otimes K_S^{-1}) &= c_1 + 2(c_1) = 3c_1, \\ \text{and } c_2(T_S \otimes K_S^{-1}) &= c_2 + c_1(c_1) + (c_1)^2 = c_2 + 2c_1^2 \end{aligned}$$

gives

$$\begin{aligned} \int_S \frac{1}{e(N^{\text{vir}})} &= (-2)^{-P_2} \int_S \frac{(1 - c_1 + c_2)(4 - 6c_1 + c_2 + 2c_1^2)}{1 - 3c_1 + c_2 + 2c_1^2} \\ &= (-2)^{-P_2} \int_S (4 - 10c_1 + 5c_2 + 8c_1^2)(1 + 3c_1 - c_2 - 2c_1^2 + 9c_1^2) \\ (8.27) \quad &= (-2)^{-\chi(K_S^2)} (c_2 + 6c_1^2). \end{aligned}$$

Remark. Finally we note the above calculations can be generalised further to the rank r case, where the “vertical” component of the \mathbb{C}^* -fixed moduli space

$$S^{[1,1,\dots,1]} \cong S$$

parameterises \mathbb{C}^* -fixed torsion sheaves

$$(\pi^* I_p) \otimes \mathcal{O}_{rS}, \quad p \in S.$$

The corresponding Higgs pairs have rank r , determinant $K_S^{-r(r-1)/2}$ and $c_2 = r + r(r-1)(r-2)(3r-1)/24$. A similar analysis to the above shows that when $g \geq 2$ the invariant is

$$\begin{aligned} &\frac{(-1)^{P_2+\dots+P_r}}{r^{P_r}} \left(\prod_{i=1}^{r-1} i^i \right)^{g-1} \times \\ &\left[r c_1(S)^2 \left(r - 1 - 2(r-1) \sum_{i=1}^{r-1} \frac{1}{i} + 2r \left(\sum_{i=1}^{r-1} \frac{1}{i} \right)^2 \right) + c_2(S) \right]. \end{aligned}$$

8.7. One mixed term. Finally we compute the contribution from the other component with $c_2 = 3$, namely the nested Hilbert scheme

$$S^{[2,1]} = \{ Z_1 \subset Z_0 \subset S : |Z_0| = 2, |Z_1| = 1 \}.$$

The torsion sheaf \mathcal{E} corresponding to $Z_1 \subset Z_0 \subset S$ (with ideal sheaves $I_1 \supset I_0$ respectively) is an extension

$$(8.28) \quad 0 \longrightarrow \iota_* I_1 \otimes K_S^{-1} \mathfrak{t}^{-1} \longrightarrow \mathcal{E} \longrightarrow \iota_* I_0 \longrightarrow 0.$$

Here $\iota: S \hookrightarrow X$ is the zero section, and multiplication by the tautological section of π^*K_S vanishing on $S \subset X$ acts on \mathcal{E} by the inclusion $\iota_*I_0 \subset \iota_*I_1$. From (8.28) we see that *in equivariant K-theory*, the class of $R\mathrm{Hom}(\mathcal{E}, \mathcal{E})[1]$ is the same as that of

$$\begin{aligned} & -R\mathrm{Hom}(\iota_*I_0, \iota_*I_0) - R\mathrm{Hom}(\iota_*I_1, \iota_*I_1) \\ & - R\mathrm{Hom}(\iota_*I_1 \otimes K_S^{-1}, \iota_*I_0)\mathfrak{t} - R\mathrm{Hom}(\iota_*I_0, \iota_*I_1 \otimes K_S^{-1})\mathfrak{t}^{-1}. \end{aligned}$$

The identity $R\mathrm{Hom}_X(\iota_*A, \iota_*B) = R\mathrm{Hom}_S(A, B) \oplus R\mathrm{Hom}_S(A, B \otimes K_S)\mathfrak{t}[-1]$ makes this

$$\begin{aligned} & -\langle I_0, I_0 \rangle + \langle I_0, I_0 \otimes K_S \rangle \mathfrak{t} - \langle I_1, I_1 \rangle + \langle I_1, I_1 \otimes K_S \rangle \mathfrak{t} \\ & - \langle I_1, I_0 \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_0 \otimes K_S^2 \rangle \mathfrak{t}^2 - \langle I_0, I_1 \otimes K_S^{-1} \rangle \mathfrak{t}^{-1} + \langle I_0, I_1 \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the K-theory class of $R\mathrm{Hom}_S(\cdot, \cdot)$.

To get the $SU(2)$ deformation-obstruction theory of Theorem 6.1 we remove a copy of $H^*(\mathcal{O}_S)$ from $\langle I_0, I_0 \rangle + \langle I_1, I_1 \rangle$ (via trace), and a copy of $H^*(K_S)\mathfrak{t}$ from $\langle I_0, I_0 \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_1 \otimes K_S \rangle \mathfrak{t}$. In particular the fixed part of the obstruction theory has the same K-theory class as

$$(8.29) \quad -\langle I_0, I_0 \rangle - \langle I_1, I_1 \rangle_0 + \langle I_0, I_1 \rangle,$$

(where the suffix 0 denotes trace-free), and the virtual normal bundle has the same K-theory class as

$$(8.30) \quad \begin{aligned} & \langle I_0, I_0 \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_1 \otimes K_S \rangle_0 \mathfrak{t} \\ & - \langle I_1, I_0 \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_0 \otimes K_S^2 \rangle \mathfrak{t}^2 - \langle I_0, I_1 \otimes K_S^{-1} \rangle \mathfrak{t}^{-1}. \end{aligned}$$

More generally this works over all of $S^{[2,1]}$ instead of at a single point. Let \mathcal{E} denote the universal sheaf on $X \times S^{[2,1]}$, let I_i denote the universal ideal sheaves on $S \times S^{[2,1]}$, and let π_X, π_S be the projections from these spaces to $S^{[2,1]}$. Then the (Serre dual) obstruction theory of $S^{[2,1]}$ of Theorem 6.1,

$$(E^\bullet)^\vee := R\mathcal{H}om_{p_X}(\mathcal{E}, \mathcal{E})_\perp[1]$$

has a class in $K^{\mathbb{C}^*}(S^{[2,1]})$ which is given by the same formula when we use $\langle \cdot, \cdot \rangle$ to denote the K-theory class of $R\mathcal{H}om_{\pi_S}(\cdot, \cdot)$. Thus (8.29) is $(E^\bullet)^{\vee, \mathrm{fix}}$ and (8.30) is $(E^\bullet)^{\vee, \mathrm{mov}}$.

We use the notation

$$\begin{array}{ccccc} S^{[2,1]} = \mathrm{Bl}_{\Delta_S}(S \times S) & \xrightarrow{p} & S^{[2]} \\ & \swarrow \pi_1 \quad \downarrow \mathrm{Bl} \quad \searrow \pi_2 & \\ S_1 & S_1 \times S_2 & S_2 \end{array}$$

for $S^{[2,1]} \cong \mathrm{Bl}_{\Delta_S}(S \times S)$. Here $S^{[2]}$ parameterises Z_0 , while $S_1 = S$ parameterises Z_1 . The support of I_{Z_1}/I_{Z_0} is a single point parameterised by S_2 . We explain and justify these claims as follows.

Consider S_1 (respectively S_2) as parameterising points $Z_1 \subset S$ (respectively $Z_2 \subset S$). There are corresponding universal points $\mathcal{Z}_i \subset S_i \times S$. Pulling them back by $\pi_1 \times \text{id}_S$, $\pi_2 \times \text{id}_S$ we get (by a small abuse of notation) universal points

$$\mathcal{Z}_1, \mathcal{Z}_2 \subset \text{Bl}_{\Delta_S}(S \times S) \times S.$$

Set

$$\mathcal{Z}_0 = \mathcal{Z}_1 \cup \mathcal{Z}_2 \subset \text{Bl}_{\Delta_S}(S \times S) \times S.$$

Since $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is a divisor in both \mathcal{Z}_i (it is a copy of the exceptional divisor $E \subset \text{Bl}_{\Delta_S}(S \times S)$) it follows that \mathcal{Z}_0 is a *flat* family of subschemes of S over $\text{Bl}_{\Delta_S}(S \times S)$. Together with the subfamily $\mathcal{Z}_1 \subset \mathcal{Z}_0$ we get a flat family of flags of subschemes of S of lengths 1 and 2 respectively, with a classifying map $\text{Bl}_{\Delta_S}(S \times S) \rightarrow S^{[2,1]}$. To construct the inverse map, note $\mathcal{Z}_0/S^{[2,1]}$ defines a classifying map $S^{[2,1]} \rightarrow S^{[2]}$, while the subscheme $\mathcal{Z}_1 \subset \mathcal{Z}_0/S^{[2,1]}$ defines classifying map from $S^{[2,1]}$ to the universal subscheme over $S^{[2]}$, which is its double cover $\text{Bl}_{\Delta_S}(S \times S)$.

From $\mathcal{Z}_1 \cap \mathcal{Z}_2 \cong E$ we get the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{Z}_2}(-E) \rightarrow \mathcal{O}_{\mathcal{Z}_0} \rightarrow \mathcal{O}_{\mathcal{Z}_1} \rightarrow 0$$

and so

$$(8.31) \quad 0 \rightarrow I_0 \rightarrow I_1 \rightarrow \mathcal{O}_{\mathcal{Z}_2}(-E) \rightarrow 0.$$

Using this (8.29) becomes

$$\begin{aligned} & -\langle I_1, I_1 \rangle + \langle \mathcal{O}_{\mathcal{Z}_2}(-E), I_1 \rangle + \langle I_1, \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle - \langle \mathcal{O}_{\mathcal{Z}_2}(-E), \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle \\ & \quad - \langle I_1, I_1 \rangle_0 + \langle I_1, I_1 \rangle - \langle \mathcal{O}_{\mathcal{Z}_2}(-E), I_1 \rangle, \end{aligned}$$

which we can write as

$$\begin{aligned} & -\langle I_1, I_1 \rangle_0 + \langle I_1, \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle - \langle \mathcal{O}_{\mathcal{Z}_2}, \mathcal{O}_{\mathcal{Z}_2} \rangle \\ & \quad = T_{S_1} + \langle I_1, \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle - \mathcal{O} + T_{S_2} - K_{S_2}^{-1}. \end{aligned}$$

Finally, writing $I_1 = \mathcal{O} - \mathcal{O}_{\mathcal{Z}_1}$ in K-theory, we obtain

$$(8.32) \quad T_{S_1} + T_{S_2} + \mathcal{O}(-E) - \langle \mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle - \mathcal{O} - K_{S_2}^{-1},$$

where we have suppressed some obvious (flat) pullback maps.

To compute the fourth term we can work on $S_1 \times S_2$ (and later pull back to $\text{Bl}_{\Delta_S}(S \times S)$ by using Bondal-Orlov's basechange [BO, Lemma 1.3]). The universal subschemes $\mathcal{Z}_1, \mathcal{Z}_2 \subset (S_1 \times S_2) \times S$ intersect transversally in the small diagonal $\delta_S \subset S^{\times 3}$. Therefore $R\mathcal{H}om(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2}) \cong \mathcal{O}_{\delta_S} \otimes \Lambda^2 N[-2]$, where N is the normal bundle to \mathcal{Z}_1 . Pushing down to $S_1 \times S_2$ we get $\mathcal{O}_{\Delta_S} \otimes \Lambda^2 N[-2]$. By a standard Koszul resolution argument the pullback of \mathcal{O}_{Δ_S} to $\text{Bl}_{\Delta_S}(S \times S)$ has $h^0 = \mathcal{O}_E$ and $h^{-1} = \mathcal{O}_E(E) \otimes \Lambda^2 N^*$, so using $\Lambda^2 N \cong K_S^{-1}$ we get

$$(8.33) \quad \langle \mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2}(-E) \rangle = \mathcal{O}_E \otimes K_S^{-1}(-E) - \mathcal{O}_E.$$

Plugging this into (8.32) gives

$$(8.34) \quad [(E^\bullet)^{\vee, \text{fix}}] = T_{S^{[2,1]}} - \text{Ob}_{S^{[2,1]}} = T_{S_1} + T_{S_2} - \mathcal{O}_E \otimes K_S^{-1}(-E) - K_{S_2}^{-1}.$$

There is an obvious exact sequence on $S^{[2,1]} \cong \text{Bl}_{\Delta_S}(S_1 \times S_2)$,

$$0 \longrightarrow T_{\text{Bl}_{\Delta_S}(S_1 \times S_2)} \xrightarrow{\text{Bl}_*} \text{Bl}^* T_{S_1 \times S_2} \longrightarrow \frac{\text{Bl}^* N_{\Delta_S}}{\mathcal{O}_E(E)} \longrightarrow 0.$$

Since $N_{\Delta_S} \cong T_S$ has rank 2, its quotient by $\mathcal{O}_E(E)$ is $\Lambda^2 N_{\Delta_S}(-E) \cong K_S^{-1}|_E(-E)$. Therefore $T_{S_1 \times S_2} = T_{S^{[2,1]}} + K_S^{-1}|_E(-E)$ in K-theory, so (8.34) simplifies to give

$$\text{Ob}_{S^{[2,1]}} = K_{S_2}^{-1},$$

at least at the level of K-theory. And $S^{[2,1]}$ is smooth, so its virtual cycle is just the top Chern class of its obstruction bundle:

$$[S^{[2,1]}]^{\text{vir}} = c_1(S_2) \cap [S^{[2,1]}].$$

In particular

$$(8.35) \quad \int_{[S^{[2,1]}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} = \int_{S^{[2,1]}} \frac{c_1(S_2)}{e(N^{\text{vir}})}.$$

Using (8.31) we rewrite (8.30) as

$$\begin{aligned} N^{\text{vir}} &= \langle I_1, I_1 \otimes K_S \rangle \mathfrak{t} - \langle I_1, \mathcal{O}_{Z_2}(-E) \otimes K_S \rangle \mathfrak{t} - \langle \mathcal{O}_{Z_2}(-E), I_1 \otimes K_S \rangle \mathfrak{t} \\ &\quad + \langle \mathcal{O}_{Z_2}(-E), \mathcal{O}_{Z_2}(-E) \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_1 \otimes K_S \rangle_0 \mathfrak{t} - \langle I_1, I_1 \otimes K_S \rangle \mathfrak{t} \\ &\quad + \langle I_1, \mathcal{O}_{Z_2}(-E) \otimes K_S \rangle \mathfrak{t} + \langle I_1, I_1 \otimes K_S^2 \rangle \mathfrak{t}^2 - \langle I_1, \mathcal{O}_{Z_2}(-E) \otimes K_S^2 \rangle \mathfrak{t}^2 \\ &\quad - \langle I_1, I_1 \otimes K_S^{-1} \rangle \mathfrak{t}^{-1} + \langle \mathcal{O}_{Z_2}(-E), I_1 \otimes K_S^{-1} \rangle \mathfrak{t}^{-1}. \end{aligned}$$

Cancelling, using Serre duality, and substituting $[I_1] = [\mathcal{O}] - [\mathcal{O}_{Z_1}]$ gives

$$\begin{aligned} &\left(-\mathcal{O}(E) + \langle \mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-E) \rangle^\vee + \langle \mathcal{O}_{Z_2}, \mathcal{O}_{Z_2} \rangle^\vee + \langle I_1, I_1 \rangle_0^\vee \right) \mathfrak{t} \\ &\quad + \left(\langle I_1, I_1 \otimes K_S^2 \rangle - K_{S_2}^2(-E) + \langle \mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-E) \otimes K_S^2 \rangle \right) \mathfrak{t}^2 \\ &\quad \left(-\langle I_1, I_1 \otimes K_S^{-1} \rangle + K_{S_2}^{-2}(E) - \langle \mathcal{O}_{Z_2}(-E), \mathcal{O}_{Z_1} \otimes K_S^{-1} \rangle \right) \mathfrak{t}^{-1}. \end{aligned}$$

Substituting in (8.33) yields

$$\begin{aligned} &\left(-\mathcal{O}(E) - K_S|_E(2E) + \mathcal{O}_E(E) + \mathcal{O} - \Omega_{S_2} + K_{S_2} - \Omega_{S_1} \right) \mathfrak{t} \\ &\quad + \left(\mathcal{O}^{\oplus P_2} - T_{S_1} \otimes K_{S_1}^2 - K_{S_2}^2(-E) + K_S|_E(-E) - K_S^2|_E \right) \mathfrak{t}^2 \\ &\quad + \left(T_{S_1} \otimes K_{S_1}^{-1} - \mathcal{O}^{\oplus P_2} + K_{S_2}^{-2}(E) + K_S^{-1}|_E(2E) - K_S^{-2}|_E(E) \right) \mathfrak{t}^{-1}. \end{aligned}$$

Finally this can be rewritten

$$\begin{aligned}
 N^{\text{vir}} = & - \left(K_{S_i}(2E) - K_{S_i}(E) + \Omega_{S_1 \times S_2} - K_{S_2} \right) \mathfrak{t} \\
 & - \left(T_{S_1} \otimes K_{S_1}^2 + K_{S_2}^2 - K_{S_j}(-E) + K_{S_j}(-2E) \right) \mathfrak{t}^2 + (\mathfrak{t}^2)^{\oplus P_2} \\
 & + \left(T_{S_1} \otimes K_{S_1}^{-1} + K_{S_2}^{-2} + K_{S_k}^{-1}(2E) - K_{S_k}^{-1}(E) \right) \mathfrak{t}^{-1} - (\mathfrak{t}^{-1})^{\oplus P_2}.
 \end{aligned}$$

for any $i, j, k \in \{1, 2\}$. Now we use $e(F \otimes \mathfrak{t}^w) = (wt)^r c_{1/wt}(E)$ for any complex of rank r , and note that N^{vir} has rank -3 , so only terms in \mathfrak{t}^0 contribute to the integral (8.35). We may therefore set $t = 1$ to give

$$\begin{aligned}
 \int_{S^{[2,1]}} & \frac{c_1(S_2) \cdot (1 + 2[E] - c_1(S_i))}{(1 + [E] - c_1(S_i))(1 - c_1(S_2))} \\
 & \cdot \frac{(1 - c_1(S_1) + c_2(S_1))(1 - c_1(S_2) + c_2(S_2))}{(1 - \frac{1}{2}c_1(S_j) - \frac{1}{2}[E])} \\
 & \cdot \frac{2^3(1 - \frac{3}{2}c_1(S_1) + \frac{1}{4}c_2(S_1) + \frac{1}{2}c_1(S_1)^2)(1 - c_1(S_2))}{(-1)^3(1 - 3c_1(S_1) + c_2(S_1) + 2c_1(S_1)^2)(1 - 2c_1(S_2))} \\
 & \cdot \frac{(1 - \frac{1}{2}c_1(S_j) - [E])(1 - c_1(S_k) - [E])(-1)^{P_2}}{(1 - c_1(S_k) - 2[E])2^{P_2}}.
 \end{aligned}$$

Multiplying out gives many terms which we group by their power of $[E]$. The $[E]^0$ terms can be easily integrated on $S_1 \times S_2$. The $[E]^1$ terms give zero since they are integrals on $E \cong \mathbb{P}(T_S) \rightarrow \Delta_S$ of Chern classes pulled back from Δ_S . The terms with $[E]^{\geq 2}$ can be evaluated using the Grothendieck formula $[E]|_E^2 - c_1(S) \cdot [E]|_E + c_2(S) = 0$ on the projective bundle $E \cong \mathbb{P}(T_S) \rightarrow \Delta_S$. Pushing down to Δ_S using the projection formula — and the fact that $[E]|_E \in H^2(E)$ pushes down to $-1 \in H^0(\Delta_S)$ — gives

$$\int_{S^{[2,1]}} [E]^2 c_1^2(S) = -c_1(S)^2 = \int_{S^{[2,1]}} [E]^3 c_1(S).$$

After much cancellation the final result is the following.

Proposition 8.36.

$$\int_{[S^{[2,1]}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} = (-2)^{-P_2(S)} c_1(S)^2 \left(-12c_1(S)^2 - 2c_2(S) + 62 \right).$$

8.8. Comparison with Vafa-Witten prediction. We can rewrite the previous formula in the notation of [VW] as

$$(8.37) \quad (-2)^{-\nu-g+1} (g-1) (-24\nu - 10g + 72),$$

where $\nu = \chi(\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + c_2(S))$ and $g = c_1(S)^2 + 1$ is the canonical genus. Adding (8.37) to the horizontal (8.24) and vertical (8.27) contributions gives the full generating series up to degree 3:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \text{VW}_{2,K_S,n}(S)q^n &= \\ (8.38) \quad & (-2)^{-\nu-g+1} \left[1 - 2(g-1)q + (2(g-1)(g-3) + 12\nu)q^2 \right. \\ & \left. - \frac{4}{3}(g-1)(18\nu + g^2 - 8g + 9)q^3 \right] + O(q^4). \end{aligned}$$

On the other hand, the second term on the first line of [VW, Equation 5.38] is, in their notation,

$$(8.39) \quad \left(\frac{1}{4}G(q^2) \right)^{\nu/2} \left(\frac{\theta_1}{\eta^2} \right)^{1-g},$$

where by [VW, Equation 5.16],

$$\begin{aligned} G(q) &= q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24}, \\ \theta_1(q) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2} = 2q^{1/4} \sum_{j=0}^{\infty} q^{j(j+1)}, \\ \frac{1}{\eta(q)} &= q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \end{aligned}$$

Therefore (8.39) is $q^{(1-g)/6-\nu}$ times by

$$2^{-\nu+1-g} \prod_{n=1}^{\infty} (1 - q^{2n})^{-12\nu} (1 - q^n)^{2g-2} \left(\sum_{j=0}^{\infty} q^{j^2+j} \right)^{1-g}.$$

Ignoring terms of $O(q^4)$ this is

$$2^{-\nu+1-g} (1 - q^2)^{-12\nu} (1 - q)^{2g-2} (1 - q^2)^{2g-2} (1 - q^3)^{2g-2} (1 + q^2)^{1-g}$$

which can be expanded as

$$\begin{aligned} 2^{-\nu+1-g} & \left(1 - (2g-2)q + \frac{(2g-2)(2g-3)}{2}q^2 - \frac{(2g-2)(2g-3)(2g-4)}{3!}q^3 \right) \\ & \times (1 + 12\nu q^2)(1 - (2g-2)q^2)(1 - (2g-2)q^3)(1 - (g-1)q^2). \end{aligned}$$

Multiplying out, this is

$$\begin{aligned} 2^{-\nu+1-g} & \left[1 - 2(g-1)q + (12\nu + 2(g-1)(g-3))q^2 \right. \\ & \left. - \frac{4}{3}(g-1)(18\nu + g^2 - 8g + 9)q^3 \right] + O(q^4), \end{aligned}$$

which agrees perfectly with (8.38) up to q^3 . We find this completely extraordinary: while Vafa and Witten do briefly consider a nonzero Higgs field in [VW, Equation 2.70], it is on a bundle rather than a sheaf. The components we have calculated with in this section consist entirely of non-locally-free sheaves not considered at all in [VW]. The physics reasoning (“cosmic strings”) used to derive [VW, Equation 2.70] is still much more powerful than our tools more than 20 years on.

The first term of [VW, Equation 5.38] seems to be the $\det E = \mathcal{O}_S$ contribution to the generating series, to which we plan to return in the future. The other terms in their equation are what Göttsche and Kool [GK] conjecture to be the contribution of the “good” component \mathcal{M}_L (1.9) of the \mathbb{C}^* -fixed locus — i.e. the virtual signed Euler characteristic (7.3) of the moduli space of stable sheaves on S with determinant L . Using Mochizuki’s work [Mo] they prove a universality result similar to those in [EGL, GNY]: this signed virtual Euler characteristic of \mathcal{M}_L is a universal expression in 7 topological constants on any surface S with $b_1(S) = 0$ and $p_g(S) > 0$. Since it is universal, the expression can be calculated on toric surfaces, which they do by torus localisation and computer calculation.²³ The result indeed reproduces the other terms of [VW, Equation 5.38] for powers $q^{c_2(E)}$ of q up to $c_2(E) = 30$.

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²³In fact they do even better, computing the (virtual) χ_y -genus refinement of the virtual Euler characteristic.

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